Introducing the difficulty of implementing alternatives in the multiple criteria decision problems

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***Abstract*—In this paper we propose methods that can help the decision-makers to find a compromise between willingness to do and ability to do by introducing the difficulty considerations in the multiple criteria decision analysis problems. Two problems are considered: ranking alternatives and improving existing solution. Usually, in the classical approaches of multiple criteria decision analysis, only the degree of satisfaction is considered to compare alternatives. However, sometimes a good alternative is difficult to implement by a decision-maker even if he spends necessary cost for it. First we give the definition of the concept of difficulty function, then we show how to introduce it in decision problems using operators based on fuzzy measures. This allows us to consider interactions between criteria under two aspects: 1) the overall satisfaction resulting from the simultaneous satisfaction or not of certain criteria; 2) the overall difficulty resulting from the difficulty or not of satisfying certain criteria simultaneously. After that, we present two examples of the difficulty function assessment in the case of a non-linear model. Finally, we propose an illustration concerning the problem of managing the students effort when improving their scores on a set of subjects. This illustration focus on the extension of the concept of worth index which quantifies the gain of improvement related to a subset of objectives when it is difficult to improve all the objectives simultaneously.**

***Index Terms*—multiple criteria decision analysis, fuzzy measure, Choquet integral, preference modelling, difficulty modelling.**

I. INTRODUCTION

The aim of Multiple Criteria Decision Analysis (MCDA) methods is to facilitate to the decision-maker (DM) the difficult task of comparing a set of alternatives when several point of view (criteria) should be tacking into account. In this work we focus on the MCDA methods derived from Multi-Attribute Value Theory (MAVT) that are based on a scoring procedure using a value function representing the DM preferences.

More precisely, let us consider a set of alternatives $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ and a set of attributes N. For each $i \in N$, the i^{th} attribute measures the extend to which an objective is satisfied and takes its values in a space X_i. Each alternative in $\mathcal{A}$ is associated with the vector of its evaluation on the attributes of N. To simplify notations, we consider $\mathcal{A} \subseteq X = \prod_{i \in N} X_i$. The DM has preferences over elements of X but generally it is difficult to express them. Let us consider that $(X, \succeq)$ represents the preferences of the DM over X where $\succeq$ represents a binary relation such that for $x, y \in X$, $x \succeq y$ means that the DM prefers x to y. The aim of MAVT [1] is to find a value function $v : X \to \mathbb{R}^+$ that represents these preferences such that:

$$\forall x, y \in X, \;\; x \succeq y \Leftrightarrow v(x) \geq v(y). \tag{1}$$

When the binary relation $\succeq$ is a *weak order* and respects the *weak preference independence* of the attributes [2], the authors of [3] state that:

$$\forall x \in X, \;\; v(x) = F(v_1(x_1), ..., v_n(x_n)) \tag{2}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ increases in all its arguments and $v_i : X_i \to \mathbb{R}^+$ is a value function that represents the DM's preferences on attribute i, $i \in N$. The *weak preference independence* assumption simplifies the assessment of v by assessing v_i for a single attribute independently (the other attributes are fixed at arbitrarily selected values) [2]. An interesting candidate for the operator F is the Choquet integral [4] related to a fuzzy measure [5] that allows modelling interactions between attributes [6] [7] and is a generalisation of numerous well-known operators such as the weighted average operator, the ordered weighted average, etc.

When v_i, $i \in N$ and F are known, one can determine a value $v(a)$ for each alternative $a \in \mathcal{A}$. Then, we can rank the alternatives in $\mathcal{A}$ as follows:

$$v(a_{\sigma(1)}) \leq v(a_{\sigma(2)}) \leq \ldots \leq v(a_{\sigma(m)})$$

$$a_{\sigma(1)} \preceq a_{\sigma(2)} \preceq \ldots \preceq a_{\sigma(m)}$$

where σ is a permutation over the set $\{1, 2, \ldots, m\}$.

The value $v(a)$ quantifies the extend to which the alternative a satisfies all the objectives of the DM. When a has a higher value $v(a)$ than another alternative b this does not signify in all situations that a is more attractive for the DM than b. Indeed, for example, if the implementation of the solution a requires to move from the actual solution a_0 to a and this move is economically or technically more difficult than the move from

a_0 to b, b could then be more attractive for the DM. Consequently, we think that the pair constitutes by the satisfaction of an alternative and its difficulty of implementation should be simultaneously taken into account to rank the alternatives of $\mathcal{A}$.

While the economic difficulty analysis decides whether or not (”go” or ”no go” decision), the project objectives can be achieved in the given time given the costs constraint, the analysis of technical difficulty is less well defined. It tends to prove that a solution can be developed taking into account the available technology and the resources required. The authors of [8], distinguish several meanings of the technical difficulty analysis: the analysis of the acceptable or unacceptable consequences of alternatives, the evaluation of the viability of a solution under boundary conditions, the selection among creative ideas of solutions satisfying the operational conditions. In this paper, the meaning of difficulty is close to latest meaning related to operational constraints. The difficulty analysis considered in the paper concerns the quantification of the extend to which it is difficult to move from the actual solution a_0 to an alternative $a \in \mathcal{A}$. Few works address the problem of ranking alternatives in a multiple criteria decision analysis problems while integrating into the decision model difficulty considerations. For example, the authors [9] consider the difficulty to move between each pair of alternatives in $\mathcal{A}$ which requires a large amount of information that may be unavailable.

A more general problem is when the set of alternative $\mathcal{A}$ is infinite. This is a multi-objective optimisation problem that consists in finding the optimal improvement of a_0, maximizing v, while respecting the difficulty constraints. Some works dealing with this problem try to determine which subobjectives to improve first in order to guarantee the maximum gain [10] [11] [12] [13] [14] [15]. In this works the difficulty considered is related either to the cost of improvement or probability/possibility of realisation of such an improvement.

In this paper we propose to introduce the assessment of the alternatives' difficulty in the previous MCDA problems: the problem of ranking alternatives and the problem of improving an actual solution. More, precisely, we propose to represent the difficulty of implementing an alternative using a real function that quantifies the difficulty to move from the actual solution to it. Then we show how to build such a function based on a fuzzy measure.

The paper is organised as follows. In the second section we introduce the difficulty function. Then in the third section we introduce the difficulty function in the ranking problem and in the fourth section we introduce it in the improvement problem. In section five we give an illustration of the proposed approach concerning the problem of improving the results of a student.

II. Difficulty function

A. Introductory example

To explain the notion of difficulty introduced in this paper, let us start with a simple example concerning a candidate student who wants to improve his chances of being admitted in a prestigious secondary school or university. The student's marks in the first half of the year are presented in Tab. I. The student knows that he should obtain an average greater

TABLE I
Student's marks

Subject	Mathematics	Physics	literature
test score x_i	8	10	12
linear model coefficient ω_i	$3/8$	$3/8$	$2/8$

or equal to 12, i.e. $\sum_{i=1}^{3} \omega_i \; x_i \geq 12$, in order to be accepted by his preferred school. Indeed, his preferred school uses a linear model to evaluate the students. Knowing that, what are the subjects' scores that the student must improve to reach this total? To answer the question we need to know the capabilities of the student to improve his mark in each course. If we know that the student can improve his score in physics until 15, he can improve his score in literature until 20 and whatever he does he can not obtain more than 12 in mathematics, then the strategy of the student is to undertake actions that focus on the improvement of the score on physics and literature instead of insisting to catch up in mathematics. Note that we treat the courses separately because of the linear model used by the school, i.e. no interactions are considered.

B. Single objective difficulty

Let us consider a set of alternatives $\mathcal{A}$ included in a space X. Let also consider a starting point $x_0 \in X$. In the same way as for the preference binary relation $\succeq$, we introduce the binary relation $\trianglelefteq_{x_0}$ where for $x, y \in \mathcal{A}$:

$x \trianglelefteq_{x_0} y$ means that x is more difficult to implement than y starting from x_0.

If $\trianglelefteq_{x_0}$ is a weak order over X, we can represent it using a real function, called in this paper difficulty function, $d_{x_0} : X \to [0, 1]$ such that:

$$x \trianglelefteq_{x_0} y \text{ iff } d_{x_0}(x) \geq d_{x_0}(y) \quad (3)$$

Fig. 1 gives an example of a possible difficulty function that could correspond to the previous example of student candidate. The function shows that the student could obtain easily a score in mathematics which is below $8 = x_0$; the difficulty increase when he tries to obtain a score higher than 8 until 12; a score above 12 is completely infeasible for him.

Note that the higher is the degree of difficulty of an alternative the lower is the alternative attractive for the DM.

C. Multi-objective difficulty

Let us now consider the case of n objectives measured by n attributes with $N = \{1, \ldots, n\}$. For each $i \in N$, the i^{th} attribute takes its values in a space X_i and we denote $X = \prod_{i \in N} X_i$. Let us consider a vector start $x^0 = (x_1^0, \ldots, x_n^0) \in X$. Let us consider that $(X, \trianglelefteq_{x^0})$ represents the difficulty relation over X for the DM. We can state the same equation as (1) to define the difficulty function in the case of multi-objective

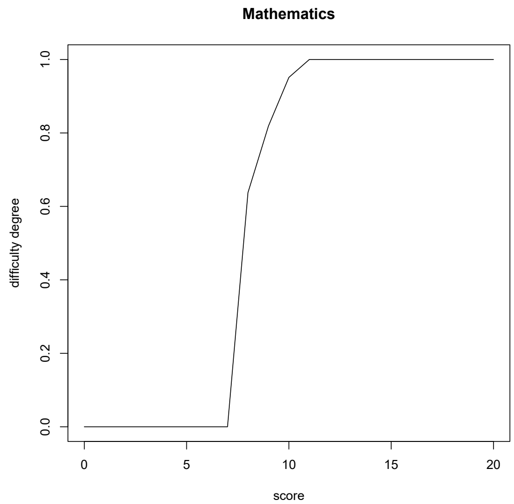

Fig. 1. Example of a difficulty function.

decision problem. Indeed, the aim is to find a difficulty function $d_{x^0} : X \to [0,1]$ that represents the difficulty of implementing alternatives in X starting from x^0 such that:

$$\forall x, y \in X, \;\; x \trianglelefteq_{x^0} y \Leftrightarrow d_{x^0}(x) \geq d_{x^0}(y). \tag{4}$$

When the binary relation $\trianglelefteq_{x^0}$ is a *weak order* and respects the *weak difficulty independence* of the attributes, d_{x^0} can be stated as follows:

$$\forall x \in X, \;\; d_{x^0}(x) = G(d_1^0(x_1), \ldots, d_n^0(x_n)) \tag{5}$$

where $G : \mathbb{R}^n \to \mathbb{R}$ increases in all its arguments and $d_i^0 : X_i \to [0,1]$ is the difficulty function related to attribute i starting from $x_i^0 \in X_i$, $\forall i \in N$. The *weak difficulty independence* assumption can be stated in the same way as *weak preference independence* and it simplifies the assessment of d_{x^0} by assessing d_i^0 for a single attribute independently (the other attributes are fixed at arbitrarily selected values). Here also we can consider the Choquet integral as a candidate for the operator G.

Finally, under the previous hypothesis, to determine d_{x^0} one has just to determine d_i^0, $i \in N$ as in subsection II-B and determine the operator G. In section IV, we give two methods to determine d_{x^0}.

III. Introducing the difficulty function in MCDA problems

Let us consider a set of attributes N. For each $i \in N$, the i^{th} attribute takes its values in a space X_i and we denote $X = \prod_{i \in N} X_i$. Let also consider a starting vector x^0. Furthermore, we consider that it exists two functions $v : X \to \mathbb{R}^+$ and $d_{x^0} : X \to \mathbb{R}^+$ verifying (1) and (4). In this section we consider that v and d_{x^0} take their values in the interval $[0,1]$.

A. Ranking alternatives

Let us consider a set of alternatives $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ such that $\mathcal{A} \subseteq X$. If one wants to rank the alternatives of $\mathcal{A}$ regarding the two objectives v and d_{x^0}, we are then faced to a multi-objective problem where v has to be maximized and d_{x^0} has to be minimized. The interesting alternatives for the DM are those belonging to the Pareto optimal set $\mathcal{P}*$ such that:

$$\mathcal{P}* = \{a \in \mathcal{A} \;:\; \nexists b \in \mathcal{A} \text{ such that } b \text{ dominate } a\} \tag{6}$$

where b dominate a means that $v(b) \geq v(a)$ and $d_{x^0}(b) \leq d_{x^0}(a)$ and at least one of the two inequalities is strict. Obviously, alternatives that reach a high degree of satisfaction require a high degree of difficulty. They are, perhaps, not the best alternatives but they will be in the Pareto optimal set since they are the best related to the first objective.

B. Improving existing solution

The difficulty function can be also introduced in the improvement problems. In particular, we consider the problem of determining the subset of attributes/objectives that one could focus the improvement on, to guarantee a minimal effort for the overall improvement to the greatest extent possible. For this aim works in [11] [10] [12] propose to associate to any coalition of attributes an index named "worth index" that quantifies for any alternative, the average of all the possible expected improvements starting from a fixed point. More precisely, let consider a value function v as in (2): $v(x) = F(v_1(x_1), ..., v_n(x_n))$, $\forall x \in X$ where $v_i : X_i \to [0,1]$. The worth index for a subset of attributes $I \subseteq N$, F is considered known, starting from x^0 denoted by $w_F(x^0, I)$ is defined by:

$$w_F(x^0, I) = \int_0^1 \frac{[F((1-\tau)v_I^0 + \tau 1\!\!1_I, v_{N\setminus I}^0) - F(v^0)]}{c(v^0, (1-\tau)v_I^0 + \tau 1\!\!1_I, v_{N\setminus I}^0)} \, d\tau \tag{7}$$

where:

- the vector $v^0 = (v_1^0, \ldots, v_n^0)$ denotes the vector of partial value functions of the vector x^0: $v_i^0 = v_i(x_i^0)$, $\forall i \in N$.
- for all $I \subseteq N$, $1\!\!1_I \in [0,1]^{|I|}$ is a vector with all components are 1.
- for two vectors $a, b \in [0,1]^n$, $(a_I, b_{N\setminus I}) \in [0,1]^n$ denotes the vector with the components of a on attributes in I, and the components of b on the other attributes.
- the quantity $\int_0^1 [F((1-\tau)v_I^0 + \tau 1\!\!1_I, v_{N\setminus I}^0) - F(v^0)] \, d\tau$ gives the mean value of the gain $[F((1-\tau)v_I^0 + \tau 1\!\!1_I, v_{N\setminus I}^0) - F(v^0)]$ only for improvement vectors on the segment from the current vector values v_I^0 (for $\tau = 0$) to the best possible improvement in I, i.e. $1\!\!1_I$ (for $\tau = 1$).
- $c(v^0, (1-\tau)v_I^0 + \tau 1\!\!1_I, v_{N\setminus I}^0)$ is the cost required for going from vector v^0 to $v^{x^0, \tau, I}$ which correspond to the cost required for going from alternative x^0 to alternative $(1-\tau)v_I^0 + \tau 1\!\!1_I, v_{N\setminus I}^0$. In [12] this cost is considered as not necessary related to monetary considerations, but related to factors they might actually be correlated with

risk appraisal, temporal requirements, resources availability, etc. This definition is close to the notion of difficulty introduced in this paper but modelled differently.

When replacing the cost function by the difficulty function, we obtain:

$$w_v(x^0, I) = \int_0^1 \frac{[F((1-\tau)v_I^0 + \tau \mathbb{1}_I, v_{N\setminus I}^0) - F(v^0)]}{d_{x^0}(v^0, (1-\tau)v_I^0 + \tau \mathbb{1}_I, v_{N\setminus I}^0)} \, d\tau \tag{8}$$

The worth index $w_v(x^0, I)$ in (8) is a weighted mean of improvements on attributes of I starting from x^0. To each improvement $[F((1-\tau)v_I^0+\tau\mathbb{1}_I, v_{N\setminus I}^0)-F(v^0)]$ is associated a weight $\frac{1}{d_{x^0}(v^0,(1-\tau)v_I^0+\tau\mathbb{1}_I,v_{N\setminus I}^0)}$.

The easy improvement will have a high weight then they will contribute with a large gain.

IV. Assessing difficulty function

Several techniques could be used to assess the difficulty function in the same way as for the value function in the multi-attribute value theory case [1] for additive and multiplicative models. In this section, we present two examples for the case of non-additive model through an aggregation procedure using Choquet integral. The assessment in the first example consists in aggregating partial difficulty functions assuming their existence. Thus, only the fuzzy measure has to be identified. In the second example, we present the disaggregation procedure applied to a non-linear model. In such case, both the partial difficulty functions and the fuzzy measure have to be identified.

In the following we recall the definitions of a fuzzy measure and the Choquet integral.

A *fuzzy measure* [17] μ over N is a set function from 2^N to $[0,1]$ such that:

1) boundary conditions: $\mu(\emptyset) = 0$ and $\mu(N) = 1$,
2) monotonicity conditions:

$$\forall K, T \subseteq N, \ K \subseteq T \Longrightarrow \mu(K) \leq \mu(T)$$

Fuzzy measures are necessary to define fuzzy integrals. The most known fuzzy integrals are Choquet integral and Sugeno integral [18].

The discrete Choquet integral of an alternative $y = (y_1, ..., y_n) \in \mathbb{R}^{+n}$ *w.r.t* a capacity μ is defined as follows:

$$C_\mu(y) = \sum_{i=1}^{n} (y_{\sigma(i)} - y_{\sigma(i-1)}) \mu(\{\sigma(i), \sigma(i+1), ..., \sigma(n)\}) \tag{9}$$

where $0 = y_{\sigma(0)} \leq y_{\sigma(1)} \leq y_{\sigma(2)} \leq ... \leq y_{\sigma(n)}$ (σ is a permutation over N).

In the case of difficulty function (9) becomes as follows:

$$\forall x \in X, \quad d_{x^0}(x) = C_\mu(d_1^0(x_1), \ldots, d_n^0(x_n))$$

A. Direct assessment of a fuzzy measure

Let us consider the introductory example of subsection II-A where a student has the scores x_i^0 obtained in a first test (see Table II). Suppose that the student should improve his scores to obtain the scores x_i, the difficulty values associated are presented in Table II.

TABLE II
STUDENT'S MARKS

Course	Mathematics (m)	Physics (p)	literature (l)
test score x_i^0	8	10	12
score to obtain x_i	12	14	18
difficulty value d_i^0	0.9	0.6	0.2
linear model coefficients ω_i	3/8	3/8	2/8

Suppose that the student wants to use (5) to assess the difficulty to obtain the score $\sum_{i=1}^{3} \omega_i \, x_i = 12$ by improving his first score from $x^0 = (8, 10, 12)$ to $x = (12, 14, 18)$ in the second test. Considering a Choquet integral for G in (5), we get:

$$d_{x^0}(x) = C_{\mu^d}(d_1^0(x_1), \ldots, d_n^0(x_n))$$

where μ^d is a fuzzy measure over $\{m, p, l\}$.

Suppose that the student tells us that: (r_1) it is somewhat easy for him to make an effort that could improve his score in literature, the effort becomes more difficult when it concerns mathematics or physics; (r_2) the difficulty change slightly when he has to work only a scientific subject compared to when he has to work literature and a scientific subject and; (r_3) the effort is more difficult when he has to work both scientific subjects than only one. From this information, one can deduce the following constraints concerning μ^d:

$$r_1: \quad \mu^d(\{m\}) \gg \mu^d(\{l\})$$
$$r_1: \quad \mu^d(\{p\}) \gg \mu^d(\{l\})$$
$$r_2: \quad \mu^d(\{m,l\}) \sim \mu^d(\{m\})$$
$$r_2: \quad \mu^d(\{p,l\}) \sim \mu^d(\{p\})$$
$$r_3: \quad \mu^d(\{m,p\}) \gg \mu^d(\{m\})$$
$$r_3: \quad \mu^d(\{m,p\}) \gg \mu^d(\{p\})$$

The following example of μ^d (see TABLE III) satisfies the constraints r_1, r_2 and r_3.

TABLE III
FUZZY MEASURE μ^d

	$\emptyset$	$\{m\}$	$\{p\}$	$\{l\}$
μ^d	0	5/8	5/8	1/8
	$\{m,p\}$	$\{m,l\}$	$\{p,l\}$	$\{m,p,l\}$
μ^d	7.5/8	5.5/8	5.5/8	1

Thus, one can determine the difficulty to obtain the scores $(12, 14, 18)$ from the scores $(8, 10, 12)$: $d_{x^0}(12, 14, 18) = \frac{6.1}{8}$.

The fact that the student decides to make the greatest improvement in literature makes the global improvement feasible but requiring a high effort.

B. Disaggregation procedure

A disaggregation procedure [19] is a procedure which is close to techniques used in machine learning. It assumes a model for the operator to determine, e.g. weighted average, Choquet integral, and then based on information provided by the DM it tries to identify the parameters of the model fitting with this information.

We consider a disaggregation procedure based on MACBETH procedure of questioning [16] and its extension to non-additive model [4]. An important issue when aggregating value from several attributes using Choquet integral is to ensure commensurateness. For this aim, in the MACBETH procedure it is assumed that the DM is able to identify for each attribute i two reference levels:

- the reference level $\mathbf{1}_i^d$ in X_i considered as totally difficult to implement starting from x^0, even if more difficult elements could exist: $d_i(\mathbf{1}_i^d) = 1$.
- the reference level $\mathbf{0}_i^d$ in X_i considered as totally easy to implement starting from x^0, even if more easier elements could exist: $d_i(\mathbf{0}_i^d) = 0$.

Note that the exponent d in the notations $\mathbf{1}_i^d$ and $\mathbf{0}_i^d$ is used to make distinction between difficulty and satisfaction. If necessary, in case of satisfaction the notations are $\mathbf{1}_i^v$ and $\mathbf{0}_i^v$.

Let us consider a set $\mathcal{A}$ of alternatives. Each alternative $a \in \mathcal{A}$ has n partial scores on n attributes. To determine $d_{x^0}(a)$ as in (5), one need to determine μ^d and $d_i^0(a_i)$, $i \in N$.

To make as easier as possible the comparisons, the DM could be asked to compare the difficulties of close alternatives. For instance, on one hand, to obtain μ^d, the DM compares alternatives $(\mathbf{1}_I^d, \mathbf{0}_{I^c}^d)$ to $(\mathbf{1}_J^d, \mathbf{0}_{J^c}^d)$, for $I, J \subset N$, i.e. $\mu^d(I) = C_{\mu^d}(\mathbf{1}_I^d, \mathbf{0}_{I^c}^d)$. On the other hand, to obtain $d_i^0(a_i)$, the DM compares alternatives $(\mathbf{1}_1^d, ..., \mathbf{1}_{i-1}^d, a_i, \mathbf{1}_{i+1}^d, ..., \mathbf{1}_n^d)$ to $(\mathbf{1}_1^d, ..., \mathbf{1}_{i-1}^d, b_i, \mathbf{1}_{i+1}^d, ..., \mathbf{1}_n^d)$ for $i \in N$, $a, b \in \mathcal{A}$. Note that to facilitate comparisons, several attributes are fixed to the same values in both comparisons.

From these comparisons two linear programs are obtained as in MACBETH procedure where the decision variables are the parameters to determine.

Note that in the case where a human decision-maker provides these preferences, one has to consider small numbers of criteria and alternatives. More then 10 criteria requires comparing at least 55 examples, i.e., alternatives, to expect capturing interactions between criteria.

V. Case study

Let us consider an illustrative example where two entities play two different roles. On the one hand, the role of candidates is played by different student profiles aiming to integrate a high school. On the other hand, the role of DM is played by two heads of two different high schools. To simplify, we consider only three test scores concerning three different subjects: Mathematics (m); Physics (p) and; Literature (l). Each head of a high school has his own preferences concerning the student profiles. The head of the first high school (HS1) prefers the students with the best general average mark while the head of the second high school (HS2) prefers students with good scientific background but also with a homogeneous profile, i.e., who have not neglected to work literature.

The student who wants to maximize his chance of success to integrate the HS1, will have an interest in working hard on the subjects with the highest coefficients in order to get the best average score. In the contrast, the student who wishes to integrate HS2 will have an interest in working on the subjects where he is weak otherwise a bad score, for example 4/20, in literature is not compensable regarding the preferences of this school.

These two examples of preferences illustrate the idea that, depending on the strategy in place, the improvement decision will not be the same (the optimum will not be the same). Improvement only becomes meaningful when the objectives and priorities are made explicit through a preferences model.

In this case study we propose to help a student to decide which subjects he should improve to guaranty a large gain considering the school that he wants to integrate and his difficulties to improve his scores.

A. Preferences representation

We express the preferences model of HS1 using an additive model considering that mathematics and physics have the same weight and they are more important than literature (see TABLE IV). Concerning HS2, we consider two rules to express its preferences model. (rp_1) the difference between the preference given to a good student in mathematics and physics and the preference given to a good one only in mathematics or in physics is very small; (rp_2) preference is given to students who are both good in science and in literature. An example μ^v of fuzzy measure respecting these rules is given in TABLE V.

TABLE IV
PREFERENCES MODEL OF HS1

	m	p	l
ω_i	3/8	3/8	2/8

TABLE V
PREFERENCES MODEL OF HS2

	$\emptyset$	$\{ma\}$	$\{p\}$	$\{l\}$
μ^v	0	3/8	3/8	2/8
	$\{m,p\}$	$\{m,l\}$	$\{p,l\}$	$\{m,p,l\}$
μ^v	4/8	6/8	6/8	1

B. Difficulty representation

Let us now consider that the students did a first test in the middle of the year and they have the second half of the year to improve their bad results. In this case the difficulty of improvement has to be taken into account. Indeed, a student will necessarily considers his ability to do better (or not) in a subject that he has already missed in the first test. Not every improvement represents the same effort for him. If having obtained $08/20$ in mathematics is already lucky for a student, he probably has no interest in betting on an improvement in his score in mathematics.

Moreover, if the grade to be raised is $02/20$ or $08/20$ in a subject that the student knows not to master completely, the challenge is not the same in two different situations: 1) the score $02/20$ could appear as a severe sanction which would not reflect his real level and gaining ten points would seem like an accessible objective. 2) the score $08/20$ is the true value of the student in the subject throughout the year, it is unlikely that the second test in this subject will allow to gain a lot of points. Thus, the difficulty of the task, the ability to do better, the probability of success, the ratio of profit to effort, also come into play in the choice of what subjects to improve.

Suppose that the scores of the first test are $x^0 = (8, 10, 12)$, an example of the three difficulty functions on each attribute are given in Fig. 2. These functions are Gaussian functions with:

$$d_i^0(x_i) = \begin{cases} 0 \text{ if } x_i \leq x_i^0 \\ exp(-\frac{(x_i - x_i^e)^2}{20}) \text{ if } x_i^0 < x_i \leq x_i^e \\ 1 \text{ if } x_i > x_i^e \end{cases}$$

where $x_1^e = 12$, $x_2^e = 15$ and $x_3^e = 20$ which means that d_i^0 increases between x_i^0 and x_i^e and reaches 1, i.e. total difficulty, when the score is 12 in mathematics, 14 in physics and 20 in literature. We consider the fuzzy measure μ^d given in TABLE III to aggregate these difficulty functions.

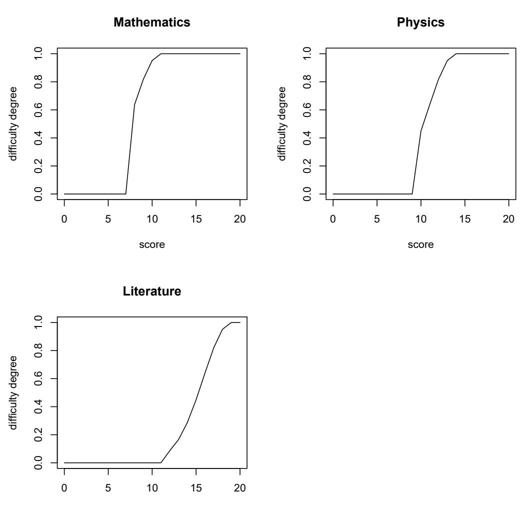

Fig. 2. Difficulty function of the three subjects.

C. The worth index

In this subsection we use the worth index defined in the subsection III-B to determine the subjects that the student should improve first to get the best overall improvement taking into account his difficulties.

Considering the preference model of HS1, we have for $I = \{m\}$:

$$\begin{aligned} w_v(x^0, \{m\}) &= [\omega_1.(20 - x_1^0)] \int_0^1 \frac{\tau \, d\tau}{d_{x^0}(x^{x^0,\tau,I})} \\ &= 4.5 \int_0^1 \frac{\tau \, d\tau}{C_{\mu^d}(d_1^0(8.(1-\tau)+20.\tau), d_2^0(10), d_3^0(12))} \\ &= \frac{4.5}{\mu^d(\{m\})}\left[\int_0^{1/3} \frac{\tau \, d\tau}{exp(-7.2\ (\tau - \frac{1}{3})^2)} + \frac{4}{9}\right] \\ &= 7.2 \int_0^{1/3} \frac{\tau \, d\tau}{exp(-7.2\ (\tau - \frac{1}{3})^2)} + 3.2 \\ &= 3.66 \end{aligned}$$

In the case of the preference model of HS2, we have:

$$\begin{aligned} w_v(x^0, \{m\}) &= \int_0^1 \frac{[C_{\mu^v}(8.(1-\tau)+20.\tau, 10, 12) - C_{\mu^v}(8, 10, 12)]}{C_{\mu^d}(d_1^0(8.(1-\tau)+20.\tau), d_2^0(10), d_3^0(12))} \, d\tau \\ &= \int_0^{1/6} \frac{3.\tau \, d\tau}{\mu^d(\{m\})\ exp(-7.2\ (\tau - \frac{1}{3})^2)} \\ &+ \int_{1/6}^{1/3} \frac{[6.\tau - 0.5] \, d\tau}{\mu^d(\{m\})\ exp(-7.2\ (\tau - \frac{1}{3})^2)} \\ &+ \int_{1/3}^{1} \frac{4.5.\tau \, d\tau}{\mu^d(\{m\})} \\ &= 3.58 \end{aligned}$$

This result shows that the student who is in this situation has more interest to improve his score in mathematics when he wants to integrate HS1 than when he wants to integrate HS2. The following tables VI and VII give the worth index of all subsets of N for the two preferences models. As we can see in these two tables, the student obtains the same and large added worth with both preference models if he improves his score in Literature. With the preference model of HS1, the student has a possibility to obtain a better added worth when improving literature than when improving all the subjects. While with the second model of HS2, he has two other possibilities: improving Mathematics and Literature or improving Physics and Literature.

TABLE VI
THE WORTH INDEX WITH THE PREFERENCES MODEL OF HS1

	$\emptyset$	$\{m\}$	$\{p\}$	$\{l\}$
$w_v(x^0, .)$	0	3.66	3.2	17.87
	$\{m, p\}$	$\{m, l\}$	$\{p, l\}$	$\{m, p, l\}$
$w_v(x^0, .)$	4.53	4.96	4.61	5.4

TABLE VII
THE WORTH INDEX WITH THE PREFERENCES MODEL OF HS2

	$\emptyset$	$\{m\}$	$\{p\}$	$\{l\}$
$w_v(x^0, .)$	0	3.58	3.62	17.87
	$\{m, p\}$	$\{m, l\}$	$\{p, l\}$	$\{m, p, l\}$
$w_v(x^0, .)$	3.61	5.37	5.62	5.26

The idea of this illustrative example, is to show that the student who has great difficulties in mathematics will not necessarily choose this subject to catch up the bad score of the first test even if the weight of Mathematics is very high.

VI. Conclusion

Concerning theoretical aspect, this paper proposes a mathematical models allowing the DM to assess alternatives not only by quantifying their degrees of satisfaction but also by quantifying the degree of difficulty to implement these alternatives. The illustration shows that the student who obtained bad scores in a first test, tends often to work hard on the subjects that he missed. However, depending on the high school that he desires to integrate, his improvement actions should not be the same. Spending a lot of time and money to improve his scores on the subjects that are very hard for him is not always the winning strategy. When a student is aware of his difficulties and of the preferences model that is used to aggregate his scores, the worth index aggregating difficulty functions could help him to decide which subjects guarantee a large gain considering his own difficulties. In the future works, risk and uncertainty considerations can also be taken into account when assessing a difficulty function.

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