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Abstract

This paper is devoted to the computation of the maximum likelihood estimates of the Mallows-Bradley-Terry ranking model parameters. The maximum likelihood method is avoid because of the normalizing constant that may involve an untractable sum with a very large number of terms. We show how to implement a Monte Carlo Maximization-Minimization algorithm to estimate the model parameters: the evaluation of the mathematical expectations involved in the log-likelihood equation is obtained by generating samples of Monte Carlo Markov chain from the stationary distribution. In addition, a simulation study for asymptotic properties assessment has been made. The proposed method is applied to analyze real life data set of the literature. The present paper is restricted to the Mallows-Bradley-Terry ranking model that does not allow for possibility of ties. This case has been studied elsewhere.

Keywords: Mallows-Bradley-Terry model, rank data, maximum likelihood method, MM-algorithm, Gibbs sampling

1. Introduction

The rankings of treatments are very common in everyday life. Gamblers want to know the underlying abilities of the horses in horse-racing competitions; companies want to know consumers' order preference on products in the market; social and political leaders want to know which candidate will get more support in the election [23]. Therefore rank data appear in many fields such as psychology, sociology, marketing, sports and econometrics.
In the literature, modeling of ranking data has received a lot of attention and many models have been proposed over the years. They could be categorized into four classes [9,10,21]: (i) paired-comparison models, (ii) distance based-models, (iii) multistage models and (iv) order-statistics models. A comprehensive discussion on these ranking models can be found in the monograph on modeling and analyzing rank data published

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by Marden [21]. Applications of the models to real life situations can be found in the literature; for example, in [6,21].

Among the four classes of the probability models for ranking data, the class of paired-comparison models may be considered as the most popular. The basic idea behind this approach is that the judges construct a legitimate ranking by starting with paired comparisons, but report their preferences only after having a set of comparisons which yield an unambiguous ranking. The resulting ranking model is referred to as the Babington Smith ranking model [21]. However the Babington Smith model involves $[q(q-1)]/2$ parameters where q is the number of treatments to be ranked . Attempts to reduce the number of the Babington Smith ranking model parameters in order to obtain more tractable and interpretable ones, yielded the Mallows-Bradley-Terry ranking model [6,21]. It is this latter that will be under consideration in this work. Severals authors [4,8,21] pointed out the main difficulty in dealing with maximum likelihood estimation (MLE) method in the computation of the normalizing constant. To get round this problem, Critchlow and Fligner (1991) have proposed to handle the Mallows-Bradley-Terry model as a Generalized Linear Model (GLM) with a log link function and a multinomial family of distribution. The normalizing constant is considered as a nuisance parameter to be estimated as the intercept of the linear predictor. This approach is until now widely used in the literature on preference data analysis, *e.g.*, in [4] or [8]. But it does not seem right to deal with the normalizing constant as a constant in so far as it depends on the model parameters.

In the present paper, the objective is to propose a new methodology to estimate the Mallows-Bradley-Terry ranking model parameters. In Section 2, we recall the Mallows-Bradley-Terry ranking model and show its properties. Section 3 consists of a maximum likelihood (ML) estimation of the Mallows-Bradley-Terry ranking model. Section 4 shows how to apply Monte Carlo Markov chain (MCMC) techniques in the Maximization-Minimization (MM) algorithm for the ML estimation of the model parameters. Section 5 describes our proposed implementation of MCMC methods to generate sample of rankings from the Mallows-Bradley-Terry ranking model. The method is validated by simulation studies in Section 6. An application of the methodology to analyze a data set available in [6] have been done in Section 7. The paper concludes with a general discussion.

2. Mallows-Bradley-Terry model for complete ranking data analysis

In paired comaparisons one considers a set of q treatments which are presented in pairs. It is assumed that the responses to the treatments may be described in terms of an underlying continuum on which the merits of the treatments can be relatively located. Let $\boldsymbol{\pi}_i$ denote the merit, an index of relative preference, of the ith treatment, $\boldsymbol{\pi}_i \geq 0, \quad \sum_{i=1}^{q} \boldsymbol{\pi}_i = 1$. The Bradley-Terry model postulates that the probability $\boldsymbol{\theta}_{ij}$ that the treatment i is preferred to treatment j is given by the formula $\boldsymbol{\theta}_{ij} = \boldsymbol{\pi}_i/(\boldsymbol{\pi}_i + \boldsymbol{\pi}_j)$.

A ranking is obtained by making independently all the pairwise comparisons until having an unambiguous ranking. The resulting ranks vector $\boldsymbol{r}$ is a finite sequence of integers of length q where each value appears once and belongs to the set of consecutive integers from 1 to q. Then the $j-$th component of $\boldsymbol{r}$, say $\boldsymbol{r}(j)$, is the rank assigned to the treatment j. The well-known Mallows-Bradley-Terry model [5] for ranking data is defined by the following probability distributions,

$$P(\boldsymbol{r};\boldsymbol{\pi}) = c(\boldsymbol{\pi}) \prod_{j=1}^{q} \boldsymbol{\pi}_j^{q-\boldsymbol{r}(j)},\ \boldsymbol{r} \in \mathcal{S}(q) \tag{1}$$

where $c(\boldsymbol{\pi})$ stands for the normalizing constant and $\mathcal{S}(q)$ is the sample space.

The identifiability constraint $\sum_{i=1}^{q} \boldsymbol{\pi}_i = 1$ on the parameters $\boldsymbol{\pi}_j$, $j = 1, 2, \dots, q$ allows re-parameterization of the model as follows: $\boldsymbol{\theta}_j = \log(\boldsymbol{\pi}_j/\boldsymbol{\pi}_q)$ for $j = 1, 2, \dots, q$. Thus $\theta_q = 0$ and this allows to have a model with $q-1$ parameters $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \cdots, \boldsymbol{\theta}_{q-1})$. The parameters $\boldsymbol{\pi}_j$, $j = 1, 2, \dots, q$ are then related to the canonical parameters $\boldsymbol{\theta}_j$, $j = 1, 2, \dots, q$ of the model by the formula $\boldsymbol{\pi}_j = \exp(\boldsymbol{\theta}_j)/\{\sum_{l=1}^{q} \exp(\boldsymbol{\theta}_l)\}$, $j = 1, 2, \dots, q$. One obtain a one-to-one correspondence between $\{\boldsymbol{\pi} \in \mathbb{R}^q : \text{each } \boldsymbol{\pi}_j > 0, \sum_{j=1}^{q} \boldsymbol{\pi}_j = 1\}$ and $\{\boldsymbol{\theta} \in \mathbb{R}^q : \boldsymbol{\theta}_q = 0\}$. The invariance property of a ML estimate can then be used. Following the reparameterization, Equation (1) can be written as

$$P(\boldsymbol{r};\boldsymbol{\theta}) = c(\boldsymbol{\theta}) \exp\left[\sum_{j=1}^{q} \{q - \boldsymbol{r}(j)\}\boldsymbol{\theta}_j\right], \boldsymbol{r} \in \mathcal{S}(q) \tag{2}$$

where $c(\boldsymbol{\theta}) = (\sum_{\boldsymbol{s}} \exp[\sum_{j=1}^{q}\{q - \boldsymbol{s}(j)\}\boldsymbol{\theta}_j])^{-1}$ denotes the normalizing constant. One obtains a curved exponential family model [16] with $q-1$ dimensions. The vector $[(q - \boldsymbol{r}(j))_{j \in \{1,2,\cdots,q-1\}}]$ is a sufficient statistic for an observed ranking $\boldsymbol{r}$ for the model. Throughout the remainder of this paper we will use the Mallows-Bradley-Terry ranking model in this form.

Severals authors pointed out the main difficulty in dealing with MLE method in the computation of the normalizing constant [21,28]. An alternative approach to overcome the computation of the normalizing constant $c(\boldsymbol{\pi})$ has been mentioned by [21] as an EM-Algorithm, but he did not give indications on how to implement this solution. This idea is pursued in what follows.

3. ML Estimation of the model parameters using a MM-algorithm

3.1. *Likelihood of the model parameters*

The log-likelihood function $l(\boldsymbol{\theta})$ of the canonical parameters $\boldsymbol{\theta}$, given an iid sample of rankings $\boldsymbol{r}_i$, $i \in \{1, 2, \cdots, n\}$ from $P(\boldsymbol{r};\boldsymbol{\theta})$, $\boldsymbol{r} \in \mathcal{S}(q)$ is

$$l(\boldsymbol{\theta}) = \sum_{i=1}^{n}\left[\sum_{j=1}^{q}\{q - \boldsymbol{r}_i(j)\}\boldsymbol{\theta}_j\right] + n\log\{c(\boldsymbol{\theta})\}. \tag{2}$$

Thus, the score vector $\nabla_{\boldsymbol{\theta}} l(\boldsymbol{\theta})$ of the log-likelihood function $l(\boldsymbol{\theta})$ at $\boldsymbol{\theta}$ is

$$\nabla_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) = \left(\sum_{i=1}^{n}\left[\mathbb{E}_{\boldsymbol{\theta}}\{\boldsymbol{s}(l)\} - \boldsymbol{r}_i(l)\right]\right)_{l \in \{1,2,\cdots,q-1\}}, \tag{3}$$

with

$$\mathbb{E}_{\boldsymbol{\theta}}(l) = \sum_{\boldsymbol{s}\in\mathcal{S}(q)} \boldsymbol{s}(l)P(\boldsymbol{s},\boldsymbol{\theta}),\ \forall l \in \{1,2,\cdots,q-1\}. \tag{4}$$

Therefore, the Hessian matrix $\nabla^2_{\boldsymbol{\theta}} l(\boldsymbol{\theta},\boldsymbol{r})$ at $\boldsymbol{\theta}$ for a ranking $\boldsymbol{r}$ is given by

$$\nabla^2_{\boldsymbol{\theta}} l(\boldsymbol{\theta};\boldsymbol{r}) = \left[-\sum_{\boldsymbol{s}\in\mathcal{S}(q)} \boldsymbol{s}(l)\boldsymbol{s}(k)P(\boldsymbol{s},\boldsymbol{\theta}) + \left\{ \sum_{\boldsymbol{s}\in\mathcal{S}(q)} \boldsymbol{s}(l)P(\boldsymbol{s};\boldsymbol{\theta}) \right\} . \left\{ \sum_{\boldsymbol{t}\in\mathcal{S}(q)} \boldsymbol{t}(k)P(\boldsymbol{t};\boldsymbol{\theta}) \right\} \right]_{k=1,2,\ldots,q-1}^{l=1,2,\ldots,q-1}.$$

There are some conditions needed to guarantee the existence and the uniqueness of the MLE $\hat{\boldsymbol{\theta}}$. These conditions are the likelihood equations and the Hessian matrix $\nabla^2_{\boldsymbol{\theta}} l(\boldsymbol{\theta};\boldsymbol{r})$ should be negative definite at $\hat{\boldsymbol{\theta}}$. Herein, they are satisfied since one can easily showed by a little algebra that the Hessian matrix $\nabla^{\mathbf{2}}{}_{\boldsymbol{\theta}} l(\boldsymbol{\theta};\boldsymbol{r})$ is negative definite. Thus the MLE $\hat{\boldsymbol{\theta}}$ exists and is unique. The paper by Yan [29] may help to find solution if the MLE does not exist.

3.2. *Introducing to MM-Algorithms*

The MM-algorithms [15] consist in a class of algorithms that includes EM-algorithm [30]. These algorithms aim to transform untractable optimization problems into tractable ones. A MM-algorithm scheme is made of the following two steps, given an estimate $\theta^{(\mathrm{m})}$ of the parameters vector $\boldsymbol{\theta}$:

(a) substitute a surrogate function $S(\boldsymbol{\theta},\boldsymbol{\theta}^{(\mathrm{m})})$ for the objective function $l(\boldsymbol{\theta})$ such that $l(\boldsymbol{\theta}) \geq S(\boldsymbol{\theta},\boldsymbol{\theta}^{(\mathrm{m})})$ and $l(\boldsymbol{\theta}^{(\mathrm{m})}) = S(\boldsymbol{\theta}^{(\mathrm{m})},\boldsymbol{\theta}^{(\mathrm{m})})$;
(b) updating the estimate of $\boldsymbol{\theta}$ by maximizing the surrogate function $S(\boldsymbol{\theta},\boldsymbol{\theta}^{(\mathrm{m})})$ with respect to $\boldsymbol{\theta}$.

It is readily seen that
if $\boldsymbol{\theta}^{(\mathrm{m+1})} = \mathrm{argmax}\{S(\boldsymbol{\theta},\boldsymbol{\theta}^{(\mathrm{m})})\}$ then $l(\boldsymbol{\theta}^{(\mathrm{m})}) \leq S(\boldsymbol{\theta}^{(\mathrm{m})},\boldsymbol{\theta}^{(\mathrm{m})}) \leq S(\boldsymbol{\theta}^{(\mathrm{m})},\boldsymbol{\theta}^{(\mathrm{m+1})}) \leq l(\boldsymbol{\theta}^{(\mathrm{m+1})})) = S(\boldsymbol{\theta}^{(\mathrm{m+1})},\boldsymbol{\theta}^{(\mathrm{m+1})})$ and therefore the MM-algorithms are monotonic. Moreover if $\nabla S(\boldsymbol{\theta}^{(\mathrm{m})},\boldsymbol{\theta}^{(\mathrm{m})}) = 0$, that is $\nabla l(\boldsymbol{\theta}^{(\mathrm{m})}) = 0$ any cluster point of a sequence defined by the recurrence $\boldsymbol{\theta}^{(m+1)} = \mathrm{argmax} S(\boldsymbol{\theta},\boldsymbol{\theta}^{(m+1)})$ is a local maximum of $l(\boldsymbol{\theta})$.

3.3. *Surrogate functions for the likelihood maximization*

In this subsection, two surrogate functions $S(\boldsymbol{\theta},\boldsymbol{\theta}')$ are proposed for the maximization of $l(\boldsymbol{\theta})$: the first is based on the ideas of Böhning and Lindsay (1988). The latter is based on the convexity of the exponential function $x \longmapsto \exp(x)$ onto $\mathbb{R}$.

Proposition 3.1. *Given a $(q-1)\times(q-1)$ matrix $\mathbf{B} = -n(q-1)^3\mathbf{I}_{q-1}$, where n denotes the sample size. Then the quadratic function $S(.;\theta')$ onto $\mathbb{R}^{q-1}$ defined by:*

$$S(\boldsymbol{\theta};\boldsymbol{\theta}') = l(\boldsymbol{\theta}') + \left[\boldsymbol{\nabla}_{\boldsymbol{\theta}'} l(\boldsymbol{\theta})\right]^t (\boldsymbol{\theta}-\boldsymbol{\theta}') + \frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\theta}')^t\,\mathbf{B}(\boldsymbol{\theta}-\boldsymbol{\theta}'), \tag{5}$$

is a surrogate function of $l(\boldsymbol{\theta})$ at $\boldsymbol{\theta}'$.

Proposition 3.2. *The real valued function* $S(.\,;\boldsymbol{\theta}')$ *onto* $\mathbb{R}^{q-1}$ *defined by,*

$$S(\boldsymbol{\theta};\boldsymbol{\theta}') = \sum_{i=1}^{n}\left[\sum_{j=1}^{q}\{q-\boldsymbol{r}_i(j)\}\boldsymbol{\theta}_j\right] - n\log\left(\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q-\boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]\right) - \frac{2n}{q(q-1)}\sum_{j=1}^{q}\left[q-\mathbb{E}_{\boldsymbol{\theta}'}(j)\right]\exp\left\{\frac{q(q-1)}{2}(\boldsymbol{\theta}_j-\boldsymbol{\theta}'_j)\right\}+1, \quad (6)$$

is a surrogate function of $l(\boldsymbol{\theta})$ *at* $\boldsymbol{\theta}'$.

The proofs of Proposition 3.1 and Proposition 3.2 are available in Appendices.

3.4. *MM-algorithms for ML estimation*

Proposition 3.1 and Proposition 3.2 lead to the following update rules

(i) $\forall l \in \{1, 2, \cdots, q-1\}$,

$$\boldsymbol{\theta}_l^{(m+1)} = \boldsymbol{\theta}_l^{(m)} + \frac{1}{(q-1)^3}\left[\mathbb{E}_{\boldsymbol{\theta}^{(m)}}(l) - \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{r}_i(l)\right], \quad (7)$$

(ii) $\forall l \in \{1, 2, \cdots, q-1\}$,

$$\boldsymbol{\theta}_l^{(m+1)} = \boldsymbol{\theta}_l^{(m)} + \frac{2}{q(q-1)}\left[\log\left\{q - \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{r}_i(l)\right\} - \log\left\{q - \mathbb{E}_{\boldsymbol{\theta}^{(m)}}(l)\right\}\right], \quad (8)$$

respectively.

Herein, the proposed MM algorithm converges like in [14]. Hereafter, we shall consider the update rules given by Equation (8). In our experimentations, it seems that the number of iterations to convergence is weaker than the others.

4. Incorporating MCMC Step in MM-algorithms for the ML-estimation

The computations of the mathematical expectations, namely Equation (4) are required and involved sums with a huge number of terms. There is no closed form analytical expression for the calculation of these mathematical expectations. This problematical computation may be overcome by approximating these expectations by their empirical means $(1/N)\sum_{k=1}^{N}\boldsymbol{r}_k(j)$ where $(\boldsymbol{r}_k)_{k=1:N}$ is a sample of rankings generated from $P(\boldsymbol{r};\boldsymbol{\theta})$, $\boldsymbol{r} \in \mathcal{S}(q)$. Such a sample may be obtained by using appropriate MCMC procedure.

4.1. *Monte Carlo MM-algorithms*

The MM-algorithm is defined by the following steps:

(i) Generate a sample of Markov chain rankings $\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_N$ from the stationary distribution $P(\boldsymbol{s};\boldsymbol{\theta})$, $\boldsymbol{s} \in \mathcal{S}(q)$ for the current values of the parameter vector $\boldsymbol{\theta}^{(m)}$,

(ii) Evaluate the estimation $\mathbb{E}_{N,\boldsymbol{\theta}^{(m)}}$ of $\mathbb{E}_{\boldsymbol{\theta}^{(m)}}$,

(iii) Update the components $\boldsymbol{\theta}_l^{(m)}$ of the parameter vector $\boldsymbol{\theta}^{(m)}$ such as $\forall l \in \{1, 2, \cdots, q-1\}$,

$$\boldsymbol{\theta}_l^{(m+1)} = \boldsymbol{\theta}_l^{(m)} + \frac{1}{q(q-1)}\left[\log\left\{q - \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{r}_i(l)\right\} - \log\left\{q - \mathbb{E}_{N,\boldsymbol{\theta}^{(m)}}(l)\right\}\right].$$

4.2. *Stopping criteria for the Monte Carlo MM-algorithms*

When dealing with the analytic MM-algorithms, the update rules are repeated until some numerical convergence criterion is full filled. Usually this stopping criterion is stated as the difference between two successive update values of the parameter vector is close to 0 or the gradient vectors is close to 0 with respect to some numerical tolerance. Such a stopping criterion is no more admissible in the case of the Monte Carlo MM-algorithm because of noise induced by Monte Carlo sampling. But we may consider the gradient vectors $\boldsymbol{\nabla}_{N,\boldsymbol{\theta}^{(m)}} l(\boldsymbol{\theta})$ close to 0 with respect to some numerical tolerance if the probability that this event does not occur is large enough (greater than a pre-specified threshold) for not to be by chance.

By following the idea of Kuk and Cheng (1997), a stopping criteria might be proposed, based on an overall statistic such that $S(\boldsymbol{\theta}^{(m)}) = N(\mathbb{E}_{N,\boldsymbol{\theta}^{(m)}} - \mathbb{E}_{\boldsymbol{\theta}^{(m)}})^t\, \hat{\boldsymbol{\Sigma}}^{-1}(\mathbb{E}_{N,\boldsymbol{\theta}^{(m)}} - \mathbb{E}_{\boldsymbol{\theta}^{(m)}})$, where $\hat{\boldsymbol{\Sigma}}$ is a consistent estimate of the covariance matrix of the vector $\mathbb{E}_{N,\boldsymbol{\theta}^{(m)}}$. Such an estimate can be yielded by using the batch means method [18]. To achieve this goal a sequence of sub-samples is built by aggregating b successive observations in order to overcome the problem of auto-correlation between consecutive sample elements. It is frequently used and owes its popularity to its simplicity and effectiveness. Hereafter, the following lemma describes how it works in our case [12,13].

Lemma 4.1. *Given $\boldsymbol{\mu} = (\mathbb{E}_{\boldsymbol{\theta}}(l))_{l\in\{1,2,\cdots,q-1\}}$, the mathematical expectation of the distribution $P(\boldsymbol{s};\boldsymbol{\theta})$, $\boldsymbol{s} \in \mathcal{S}(q)$ and a sample of size N of Monte Carlo rankings $\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_N$ in which ties are not possible at any step m corresponding to a parameter vector $\boldsymbol{\theta}^{(m)}$. The Monte Carlo sample of size N are split into G groups of b consecutive rankings. Assuming that $N = G * b$, the estimator of the empirical mean of any group g, $g \in \{1, 2, \ldots, G\}$ is given by,*

$$\overline{\boldsymbol{Y}}_g = \left\{\frac{1}{b}\sum_{i=(g-1)b+1}^{g*b} \boldsymbol{s}_i(l)\right\}_{l=1,2,\cdots,q-1}.$$

The empirical mean (total mean of all the G groups) is given by

$$\overline{\boldsymbol{\mu}}_N = \left\{\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{s}_i(l)\right\}_{l=1,2,\cdots,q-1} = \frac{1}{G}\sum_{g=1}^{G}\overline{\boldsymbol{Y}}_g.$$

Assuming that the size b of each group G is large enough so that the vector $\overline{\boldsymbol{Y}}_g$, $g \in \{1, 2, \cdots, G\}$are independently and approximately gaussian $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}/b)$ with expecta-

tion $\boldsymbol{\mu}$ *and covariance matrix* $\boldsymbol{\Sigma}/\mathbf{b}$. *Then* $\boldsymbol{\Sigma}$ *can be approximated by,*

$$\hat{\boldsymbol{\Sigma}} = \frac{b}{G-1}\sum_{g=1}^{G}(\overline{\boldsymbol{Y}}_g - \overline{\boldsymbol{\mu}}_N)(\overline{\boldsymbol{Y}}_g - \overline{\boldsymbol{\mu}}_N)^t.$$

Proposition 4.2. *Given* $\boldsymbol{\mu} = (\mathbb{E}_{\boldsymbol{\theta}}(l))_{l\in\{1,2,\cdots,q-1\}}$, *the mathematical expectation of the distribution* $P(\boldsymbol{s};\boldsymbol{\theta})$, $\boldsymbol{s} \in \mathcal{S}(q)$. *Let* $\mathbf{W} = \hat{\boldsymbol{\Sigma}}/b = \{1/(G-1)\}\sum_{g=1}^{G}(\overline{\boldsymbol{Y}}_g - \overline{\boldsymbol{\mu}}_N)(\overline{\boldsymbol{Y}}_g - \overline{\boldsymbol{\mu}}_N)^t$ *be the empirical covariance matrix of the random vectors* $\overline{\boldsymbol{Y}}_g$, $g \in \{1, 2, \cdots, G\}$. *The statistic* $T^2 = G(\overline{\boldsymbol{\mu}}_N - \boldsymbol{\mu})^t\mathbf{W}^{-1}(\overline{\boldsymbol{\mu}}_N - \boldsymbol{\mu})$ *is a Hotelling* T^2 *statistic with parameters* q *and* $G-1$ *denoted by* $T^2(q, G-1)$.
Then, the statistic

$$\frac{G(G-q)}{(G-1)q}(\overline{\boldsymbol{\mu}}_N - \boldsymbol{\mu})^t\mathbf{W}^{-1}(\overline{\boldsymbol{\mu}}_N - \boldsymbol{\mu}),$$

has an asymptotic F-distribution with q *and* $G-q$ *degrees of freedom.*

Using the batch means methods described above, one can propose a stopping criterion in the MM-algorithm as follows.

Proposition 4.3. *Given* $\mathbf{W} = \hat{\boldsymbol{\Sigma}}/b = \{1/(G-1)\}\sum_{g=1}^{G}(\overline{\boldsymbol{Y}}_g - \overline{\boldsymbol{\mu}}_N)(\overline{\boldsymbol{Y}}_g - \overline{\boldsymbol{\mu}}_N)^t$ *the empirical covariance matrix of the random vectors* $\overline{\boldsymbol{Y}}_g$, $g \in \{1, 2, \cdots, G\}$ *and* $f_{q-1,\,G-q+1}$ *the upper* α *quantile of the F-distribution with* $q-1$ *and* $G-q+1$ *degrees of freedom. One stop the MM-algorithm when*

$$f_{q-1,\,G-q+1} > \frac{G(G-q+1)}{(G-1)(q-1)}(\overline{\boldsymbol{\mu}}_N - \boldsymbol{\mu})^t\mathbf{W}^{-1}(\overline{\boldsymbol{\mu}}_N - \boldsymbol{\mu}).$$

In the batch means method, the estimation $\hat{\Sigma}$ of Σ is consistent [11] when G and b are chosen such that $b = \lfloor N^v \rfloor$ and $G = \lfloor N/b \rfloor$ with $v \in \mathbb{Q}$, where the notation $\lfloor v \rfloor$ for any real number v denotes the floor of v.

5. Simulation experiments

Since one have to deal with a multivariate distribution, it is natural to think of Gibbs method [2,25,26]. Given a ranking $\boldsymbol{s} = (\boldsymbol{s}(1), \boldsymbol{s}(2), \cdots, \boldsymbol{s}(q))$ of q treatments, its components $\boldsymbol{s}(j)$ are subject to the constraint $\sum_{j=1}^{q}\boldsymbol{s}(j) = q(q+1)/2$. Hence, it is not possible to use a classical Gibbs algorithm. An algorithm of independent Metropolis-Hastings [2,25,26] with uniform distribution on the sample space $\mathcal{S}(q)$, as an instrumental distribution, to generate the proposed ranking for the next state is a possible solution. Because of the large dimension of the sample space $\mathcal{S}(q)$, the convergence towards the stationary distribution may be difficult. Herein, we propose a Generalized Gibbs sampler algorithm inspired by the idea of Diaconis (1988) and Lange (2000) to generate a trajectory of a Markov chain from the stationary distribution $P(\boldsymbol{s};\boldsymbol{\theta})$, $\boldsymbol{s} \in \mathcal{S}(q)$.

5.1. *Generalized Gibbs sampler*

The notation $\tau(a,b)$ denotes the transposition on the set $\{1,2,\cdots,q\}$, that permutes the integers a and b with $a \neq b$. Since a ranking $\boldsymbol{r} = (\boldsymbol{r}(1), \boldsymbol{r}(2), \cdots, \boldsymbol{r}(j), \cdots, \boldsymbol{r}(q))$ is a vector of integers where $\boldsymbol{r}(j)$ denotes the rank assigned to the treatment j, we define

$$\tau(a,b)\boldsymbol{r} = \boldsymbol{s} \Longleftrightarrow \begin{cases} \boldsymbol{s}(j) = \boldsymbol{r}(j) & \text{if } j \neq \boldsymbol{r}^{-1}(a),\ j \neq \boldsymbol{r}^{-1}(b) \\ \boldsymbol{s}(\boldsymbol{r}^{-1}(a)) = b & \\ \boldsymbol{s}(\boldsymbol{r}^{-1}(b)) = a. & \end{cases}$$

We propose the following algorithm inspired by the idea of Diaconis (1988) and Lange (2000) to generate a trajectory of a Markov chain from the stationary distribution $P(\boldsymbol{s};\boldsymbol{\theta})$, $\boldsymbol{s} \in \mathcal{S}(q)$.

Algorithm 1 Generalized Gibbs sampler.

Initializing $\boldsymbol{r}$: arbitrarily choose a ranking $\boldsymbol{r}$ in the $\mathcal{S}(q)$ as the initial state of the Markov chain;
Repeat until convergence;
BEGIN
1. Randomly draw a couple of distinct integers (a,b) from the set $\{1,2,\cdots,q\}$ with uniform distribution;
2.IF $U(0,1) < \frac{P(\tau(a,b)\boldsymbol{r};\boldsymbol{\theta})}{P(\boldsymbol{r};\boldsymbol{\theta})}$ then $\boldsymbol{r} = \tau(a,b)\boldsymbol{r}$ otherwise $\boldsymbol{r} = \boldsymbol{r}$;
END.

On obtain a trajectory of a Markov chain with transition probabilities $q(\boldsymbol{r},\boldsymbol{s})$ from a ranking $\boldsymbol{r}$ to a ranking $\boldsymbol{s}$ defined as follows:

$$q(\boldsymbol{r},\boldsymbol{s}) = \begin{cases} \frac{2}{q(q-1)} & \text{if } \exists(a,b),\ \boldsymbol{s} = \tau(a,b)\boldsymbol{r},\ \frac{P(\tau(a,b)\boldsymbol{r};\boldsymbol{\theta})}{P(\boldsymbol{r};\boldsymbol{\theta})} \geq 1 \\ \frac{2}{q(q-1)} \frac{P(\tau(a,b)\boldsymbol{r};\boldsymbol{\theta})}{P(\boldsymbol{r};\boldsymbol{\theta})} & \text{if } \exists(a,b),\ \boldsymbol{s} = \tau(a,b)\boldsymbol{r},\ \frac{P(\tau(a,b)\boldsymbol{r};\boldsymbol{\theta})}{P(\boldsymbol{r};\boldsymbol{\theta})} < 1 \\ \frac{2}{q(q-1)} \sum_{a,b}[1 - \frac{P(\tau(a,b)\boldsymbol{r};\boldsymbol{\theta})}{P(\boldsymbol{r};\boldsymbol{\theta})}] & \text{if } \boldsymbol{s} = \boldsymbol{r},\ \frac{P(\tau(a,b)\boldsymbol{r};\boldsymbol{\theta})}{P(\boldsymbol{r};\boldsymbol{\theta})} \geq 1 \\ 0 & \text{if } \forall(a,b),\ \boldsymbol{s} \neq \tau(a,b)\boldsymbol{r}. \end{cases}$$

6. Simulations study

In this section the use of the proposed method is demonstrated by simulations study. This method has been implemented using R [24].

We generated samples of iid rankings distributed as the probability $P(\boldsymbol{s};\boldsymbol{\theta}), \boldsymbol{s} \in \mathcal{S}(q)$ via the MCMC method described above for different values of q namely $q = 4$ and $q = 8$. For a known parameter vector $\boldsymbol{\theta}$ of the parameter space, each sample of rankings with size M is obtained by generating M independent trajectories with stationary distribution $P(\boldsymbol{s};\boldsymbol{\theta}), \boldsymbol{s} \in \mathcal{S}(q)$ until a stopping time.

The Monte Carlo samples of size N used to estimate the mathematical expectations are obtained by subtracting N observations of a single trajectory. The burn-in time is a function of q, say $q^\beta \log(q)$ where $\beta \in \mathbb{R}$.

For a known parameter vector $\boldsymbol{\theta}$, 1000 samples of rankings with sizes 100, 200 and 500 have been generated. Then, the samples are used to estimate the model parameter vector $\boldsymbol{\theta}$. The generated MCMC samples size is $N = 20000$ at each step of the MM algorithm. The burn-in time is set to $q^\beta log(q) = 100000$ which gives $\beta = 8.069$ for

Table 1. Estimates of the model parameter vector for 1000 samples of rankings with common size 100, 200 and 500 for $q = 4$: MSE (empirical mean square error)

Size	Parameter	True value	Mean	Std.dev	MSE
100	$\boldsymbol{\pi}_1$	0.088	0.082	1.345×10^{-4}	3.041×10^{-4}
	$\boldsymbol{\pi}_2$	0.481	0.492	8.026×10^{-4}	1.729×10^{-3}
	$\boldsymbol{\pi}_3$	0.150	0.144	2.326×10^{-4}	5.051×10^{-4}
	$\boldsymbol{\pi}_4$	0.282	0.283	4.820×10^{-4}	9.653×10^{-4}
200	$\boldsymbol{\pi}_1$	0.088	0.0837	7.138×10^{-5}	1.580×10^{-4}
	$\boldsymbol{\pi}_2$	0.481	0.487	3.898×10^{-4}	8.209×10^{-4}
	$\boldsymbol{\pi}_3$	0.150	0.145	1.206×10^{-4}	2.686×10^{-4}
	$\boldsymbol{\pi}_4$	0.282	0.284	2.461×10^{-4}	4.994×10^{-4}
500	$\boldsymbol{\pi}_1$	0.088	0.083	2.568×10^{-5}	7.128×10^{-5}
	$\boldsymbol{\pi}_2$	0.481	0.488	1.462×10^{-4}	3.547×10^{-4}
	$\boldsymbol{\pi}_3$	0.150	0.144	4.728×10^{-5}	1.344×10^{-4}
	$\boldsymbol{\pi}_4$	0.282	0.285	1.057×10^{-4}	2.198×10^{-4}

$q = 4$ and $\beta = 5.184$ for $q = 8$. The power v parameter appearing in the calculation of the number of groups G and the common size b of groups in the batch means method is set to $v = 0.5$, *i.e.* $G = 141$ and $b = 141$. The upper $\alpha = 0.05$ quantile $f_{q-1,G-q+1}$ of the F-distribution with $q-1$ and $G-q+1$ degrees of freedom involved in the stopping criterion of the MM-algorithm, is equal to $f_{q-1,G-q+1} = 2.670$ for $q = 4$ and $f_{q-1,G-q+1} = 2.079$ for $q = 8$.

Table 1 and Table 2 contain the estimates of the model parameter vector for $q = 4$ and $q = 8$ respectively. It can be seen that all the parameters empirical means are very close to their corresponding true parameter values. Moreover, the estimators precisions are good and increase when the sample size increases. This indicates that the model parameter vector estimates are good. Furthermore the figures, Figure C1, Figure C2, Figure C3 on the one hand, and Figure C4, Figure C5, Figure C6 on the other hand, depict the empirical quantiles of the canonical parameter vector components estimates with respect to their corresponding quantiles of the standard gaussian distribution $\mathcal{N}(0,1)$ for $q = 4$ and $q = 8$ respectively (*see* Appendice, Section C). The straight lines correspond to the reference line $y = x$. One can state that the graphs look like a straight line. This suggests that the estimations of the parameters are asymptotically gaussian.

7. Application to real data

We now use our methodology to analyze a set of ranking data from Example 3 in the paper by Critchlow and Fligner (1993), in which 32 consumers have ranked four salad dressing preparations according to tartness, with a rank of 1 being assigned to the formulation judged to be the most tart. The data are given in Table 3.

The simultaneous $(1-\alpha)$-bootstrap confidence intervals (SCI) based on the percentile bootstrap approach [22] have been calculated for the parameter vector $\boldsymbol{\theta}$. To this end, we needed the package boot [1]. The following estimates of the treatments parameters have been found: $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, 0) = (-0.096, 2.416, 0.888, 0)$ with confidence intervals $(-1.000, -0.050)$, $(2.600, 2.300)$, $(0.500, 1.000)$ at a significance level of $\alpha = 0.05$ respectively. Then, the value of merit of each salad is $\boldsymbol{\pi}_A = 0.058$, $\boldsymbol{\pi}_B = 0.721$, $\boldsymbol{\pi}_C = 0.156$ et $\boldsymbol{\pi}_D = 0.064$.

Table 2. Estimates of the model parameter vector for 1000 samples of rankings with common size 100, 200 and 500 for $q = 8$: MSE (empirical mean square error)

Size	Parameter	True value	Mean	Std.dev	MSE
100	$\boldsymbol{\pi}_1$	0.092	0.094	1.587×10^{-5}	3.482×10^{-5}
	$\boldsymbol{\pi}_2$	0.105	0.107	1.462×10^{-5}	3.080×10^{-5}
	$\boldsymbol{\pi}_3$	0.093	0.095	1.533×10^{-5}	3.231×10^{-5}
	$\boldsymbol{\pi}_4$	0.104	0.106	1.406×10^{-5}	2.973×10^{-5}
	$\boldsymbol{\pi}_5$	0.170	0.172	3.971×10^{-5}	8.124×10^{-5}
	$\boldsymbol{\pi}_6$	0.150	0.150	2.128×10^{-5}	4.285×10^{-5}
	$\boldsymbol{\pi}_7$	0.185	0.187	5.819×10^{-5}	1.249×10^{-4}
	$\boldsymbol{\pi}_8$	0.100	0.090	8.480×10^{-6}	1.246×10^{-4}
200	π_1	0.092	0.082	6.999×10^{-6}	1.850×10^{-5}
	$\boldsymbol{\pi}_2$	0.105	0.107	5.958×10^{-6}	1.426×10^{-5}
	$\boldsymbol{\pi}_3$	0.093	0.096	6.656×10^{-6}	1.841×10^{-5}
.	$\boldsymbol{\pi}_4$	0.104	0.106	6.090×10^{-6}	1.514×10^{-5}
	$\boldsymbol{\pi}_5$	0.170	0.171	1.654×10^{-5}	3.323×10^{-5}
	$\boldsymbol{\pi}_6$	0.150	0.150	1.034×10^{-5}	2.092×10^{-5}
	$\boldsymbol{\pi}_7$	0.185	0.186	2.850×10^{-5}	5.933×10^{-5}
	$\boldsymbol{\pi}_8$	0.100	0.090	4.145×10^{-6}	1.092×10^{-4}
500	π_1	0.092	0.095	2.766×10^{-6}	1.157×10^{-5}
	$\boldsymbol{\pi}_2$	0.105	0.107	2.482×10^{-6}	8.032×10^{-6}
	$\boldsymbol{\pi}_3$	0.093	0.096	3.010×10^{-6}	1.144×10^{-5}
	$\boldsymbol{\pi}_4$	0.104	0.106	2.423×10^{-6}	8.317×10^{-6}
	$\boldsymbol{\pi}_5$	0.170	0.171	7.098×10^{-6}	1.457×10^{-5}
	$\boldsymbol{\pi}_6$	0.150	0.150	4.153×10^{-6}	8.381×10^{-6}
	$\boldsymbol{\pi}_7$	0.185	0.185	8.509×10^{-6}	1.764×10^{-5}
	$\boldsymbol{\pi}_8$	0.100	0.090	1.891×10^{-6}	1.053×10^{-4}

Table 3. Salad Dressing rankings.

Rank Vectors Observed	Frequency
(1,2,3,4)	2
(2,1,3,4)	1
(2,1,4,3)	2
(2,3,1,4)	1
(3,1,2,4)	2
(3,1,4,2)	1
(3,4,2,1)	1
(4,1,2,3)	11
(4,1,3,2)	6
(4,2,1,3)	3
(4,3,1,2)	1
(4,3,2,1)	1

This gives the ranking $\tilde{\boldsymbol{r}} = (4, 1, 2, 3)$, which is the ranking most consumers have chosen in the data in Table 3. The observed rank vector contains the ranks given to the four salad dressing preparations A, B, C and D.

8. Concluding remarks

In this paper we have proposed a new methodology to estimate the Mallows-Bradley-Terry ranking model parameters. Our work is restricted to the Mallows-Bradley-Terry ranking model that does not allow for possibility of ties. The case which allows for possibility of ties has been proposed and entirely studied in [27].

A simple MM-algorithm has been proposed for the calculation of the ML estimators of the model parameter vector. The update rules inferred from the MM-algorithm require the computations of the mathematical expectations of the model probability distribution. These computations involved sums with a huge number of terms. For such high-dimensional summations, numerical summation is no longer feasible or reliable and Monte Carlo methods are more appealing. Thus, the evaluation of the mathematical expectations involved in the log-likelihood equation is obtained by generating samples of MCMC from the probability distribution. To achieve this goal, a generalized Gibbs Sampler algorithm was proposed. We also suggest a stopping criterion in the MM-algorithm based on a F-distribution test for zero gradient.

The use of the methodology has been validated by a simulation study and illustrated by an application to real life data. The proposed method works very well for large number q of treatments to be ranked. Herein, we only show the results of simulation for $q = 4$ and $q = 8$ treatments to save place. In the simulation study, for known theoretical parameter vector, samples of rankings with different sizes have been generated from the model probability distribution. Then, these samples are used to estimate the parameter vector. It results that estimates exhibit small biais and asymptotic normality. The illustration of the proposed methodology is based on an application to experimental data. For this purpose, the data from [6] have been used.

The simulation study and the example based on a real life data show that the proposed method to estimate the Mallows-Bradley-Terry model parameter vector is computationally efficient and more flexible.

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Appendix A. Proof of Proposition 3.1

Proof. The proof of Proposition 3.1 is based on the ideas of Böhning and Lindsay [3] We recall that the hessian matrix is defined by

$$\nabla^2_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) = -n\mathbf{M}\{\mathcal{S}(q)\}^t[\mathrm{diag}\{p(\boldsymbol{\theta})\} - p(\boldsymbol{\theta})p(\boldsymbol{\theta})^t]\mathbf{M}\{\mathcal{S}(q)\}$$

where the vector $p(\boldsymbol{\theta}) = \{P(\boldsymbol{s};\boldsymbol{\theta})\}_{\boldsymbol{s}\in\mathcal{S}(q)}$ and matrix $\mathbf{M}\{\mathcal{S}(q)\}$ is the $q! \times (q-1)$ with elements $\mathbf{M}\{\mathcal{S}(q)\} = \{\boldsymbol{s}_i(j)\}_{i=1,2,\cdots,q!}^{j=1,2,\cdots,(q-1)}$ where $\forall i \in \{1,2,\cdots,q!\}, \boldsymbol{s}_i \in \mathcal{S}(q)$.
It is readily seen that

$$\nabla^2_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) = -n\mathbf{V}$$

with $\mathbf{V} = \{\mathrm{Cov}_{\boldsymbol{\theta}}(l,k)\}_{k=1,2,\cdots,q-1}^{l=1,2,\cdots,(q-1)}$.
In accordance with Cauchy-Schwarz's inequality, one has:
$\forall k, l \in \{1,2,\cdots,q-1\}$,

$$\begin{aligned}|v_{kl}| = |\mathrm{Cov}_{\boldsymbol{\theta}}(l,k)| &\leq \mathrm{Var}^{1/2}_{\boldsymbol{\theta}}(l)\mathrm{Var}^{1/2}_{\boldsymbol{\theta}}(k)\\ &\leq (q-1)^2,\end{aligned}$$

because $1 \leq \boldsymbol{s}(k) \leq q, \forall k \in \{1,2,\cdots,q\}$ and then $\mathrm{Var}^{1/2}_{\boldsymbol{\theta}}\{\boldsymbol{s}(k)\} \leq q-1$.
Thus, $\forall a \in \mathbb{R}^{q-1}$,

$$\begin{aligned}{}^t a\mathbf{V}a &\leq \sum_{k=1}^{q}\sum_{l=1}^{q}|a_k||a_l||v_{kl}|\\ &\leq (q-1)^2\sum_{k=1}^{q}\sum_{l=1}^{q}|a_k||a_l|\\ &\leq (q-1)^2\left(\sum_{k=1}^{q}|a_k|\right)\left(\sum_{l=1}^{q}|a_l|\right)\\ &\leq (q-1)^2(q-1)^{1/2}(q-1)^{1/2}\left(\sum_{k=1}^{q}a_k^2\right)\\ &\leq (q-1)^3\,{}^t a\mathbf{I}_{q-1}a,\end{aligned}$$

where $\mathbf{I}_{q-1}$ is the $(q-1)\times(q-1)$ identity matrix. One infers that $\mathbf{V} \leq (q-1)^3\mathbf{I}_{q-1}$. The matrix $\mathbf{B} = -n(q-1)^3\mathbf{I}_{q-1}$ is negative definite and $\nabla^2_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) - \mathbf{B}$ is semi-positive. Therefore, the quadratic function $S(\boldsymbol{\theta};\boldsymbol{\theta}') = l(\boldsymbol{\theta}') + \{\nabla_{\boldsymbol{\theta}'} l(\boldsymbol{\theta})\}^t(\boldsymbol{\theta}-\boldsymbol{\theta}') + 1/2(\boldsymbol{\theta}-\boldsymbol{\theta}')^t\mathbf{B}(\boldsymbol{\theta}-\boldsymbol{\theta}')$ is a surrogate function of $l(\boldsymbol{\theta})$ at $\boldsymbol{\theta}'$.
Thus the MM algorithm proceeds by maximizing the fonction $S(\boldsymbol{\theta},\boldsymbol{\theta}')$ with respect to $\boldsymbol{\theta}$, giving the following algorithm

$$\begin{aligned}\boldsymbol{\theta}^{(m+1)} &= \boldsymbol{\theta}^{(m)} - B^{-1}\nabla_{\boldsymbol{\theta}^{(m)}} l(\boldsymbol{\theta})\\ &= \boldsymbol{\theta}^{(m)} + \frac{1}{(q-1)^3}\left\{\frac{1}{n}\nabla_{\boldsymbol{\theta}^{(m)}} l(\boldsymbol{\theta})\right\}.\end{aligned}$$

Finally,

$$\boldsymbol{\theta}^{(m+1)} = \boldsymbol{\theta}^{(m)} + \frac{1}{(q-1)^3}\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\mathbb{E}_{\boldsymbol{\theta}^{(m)}} - \boldsymbol{r}_{i\cdot}\right]\right\}$$

□

Appendix B. Proof of Proposition 3.2

Proof. The log-likelihood associated with a sample of rankings $\boldsymbol{r}_i$, $i \in \{1, 2, \cdots, n\}$ iid is given by

$$l(\boldsymbol{\theta}) = \sum_{i=1}^{n}\left[\sum_{j=1}^{q}\{q - \boldsymbol{r}_i(j)\}\boldsymbol{\theta}_j\right] + n\log\{c(\boldsymbol{\theta})\},$$

with $\log\{c(\boldsymbol{\theta})\} = -\log\left(\sum_{\boldsymbol{s}}\exp\left[\left\{\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}_j\right]\right)$.

As the function $x \longmapsto -\log(x)$ is convex on $]0, +\infty[$, then giving $u_0 \in]0, +\infty[$, $-\log(u) \geq -\log(u_0) - (u - u_0)/u_0$, $\forall u \in]0, +\infty[$. Thus

$$\begin{aligned}\log\{c(\boldsymbol{\theta})\} \geq & -\log\left(\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]\right) \\ & - \frac{\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}_j\right]}{\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]} + 1.\end{aligned}$$

Let's set,

$$Q(\boldsymbol{\theta}) = -\frac{\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}_j\right]}{\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]}.$$

One obtains,

$$Q(\boldsymbol{\theta}) = -\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}(\boldsymbol{\theta}_j - \boldsymbol{\theta}'_j)\right] \times P(\boldsymbol{s}; \boldsymbol{\theta}'),$$

thus,

$$\log\{c(\boldsymbol{\theta})\} \geq -\log\left(\sum_{\boldsymbol{s}} \exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]\right)$$
$$- \sum_{\boldsymbol{s}} \exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}(\boldsymbol{\theta}_j - \boldsymbol{\theta}'_j)\right] P(\boldsymbol{s};\boldsymbol{\theta}')$$
$$+ 1.$$

Let,

$$Q(\boldsymbol{s};\boldsymbol{\theta}) = \exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}(\boldsymbol{\theta}_j - \boldsymbol{\theta}'_j)\right].$$

One has, since the exponential function $x \longmapsto \exp(x)$ is convex on the real set $\mathbb{R}$,

$$Q(\boldsymbol{s};\boldsymbol{\theta}) \leq \frac{2}{q(q-1)}\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\exp\{\frac{q(q-1)}{2}(\boldsymbol{\theta}_j - \boldsymbol{\theta}'_j)\}$$

By multiplying the two members of the previous inequality by $P(\boldsymbol{s};\boldsymbol{\theta}')$, then summing, we have

$$Q(\boldsymbol{\theta}) \geq -\frac{2}{q(q-1)}\sum_{j=1}^{q}\left[q - \mathbb{E}_{\boldsymbol{\theta}'}(j)\right]\exp\{\frac{q(q-1)}{2}(\boldsymbol{\theta}_j - \boldsymbol{\theta}'_j)\}$$

leading to

$$\log\{c(\boldsymbol{\theta})\} \geq -\log\left(\sum_{\boldsymbol{s}} \exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]\right)$$
$$- \frac{2}{q(q-1)}\sum_{j=1}^{q}\left[q - \mathbb{E}_{\boldsymbol{\theta}'}(j)\right]\exp\left\{\frac{q(q-1)}{2}(\boldsymbol{\theta}_j - \boldsymbol{\theta}'_j)\right\}+1.$$

It results that,

$$l(\boldsymbol{\theta}) \geq \sum_{i=1}^{n}\left[\sum_{j=1}^{q}\{q - \boldsymbol{r}_i(j)\}\boldsymbol{\theta}_j\right] - n\log\left(\sum_{\boldsymbol{s}} \exp\left[\sum_{j=1}^{q}\left\{q - \boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]\right)$$
$$- \frac{2n}{q(q-1)}\sum_{j=1}^{q}\left[q - \mathbb{E}_{\boldsymbol{\theta}'}(j)\right]\exp\left\{\frac{q(q-1)}{2}(\boldsymbol{\theta}_j - \boldsymbol{\theta}'_j)\right\}+1.$$

It comes that the following function $S(\boldsymbol{\theta};\boldsymbol{\theta}')$ defined below is a surrogate function of the log likelihood $l(\theta)$

$$S(\boldsymbol{\theta};\boldsymbol{\theta}') = \sum_{i=1}^{n}\left[\sum_{j=1}^{q}\{q-\boldsymbol{r}_i(j)\}\theta_j\right] - n\log\left(\sum_{\boldsymbol{s}}\exp\left[\sum_{j=1}^{q}\left\{q-\boldsymbol{s}(j)\right\}\boldsymbol{\theta}'_j\right]\right)$$
$$- \frac{2n}{q(q-1)}\sum_{j=1}^{q}\left[q-\mathbb{E}_{\boldsymbol{\theta}'}(j)\right]\exp\left\{\frac{q(q-1)}{2}(\boldsymbol{\theta}_j-\boldsymbol{\theta}'_j)\right\}+1.$$

Thus, the MM alogorithm proceeds by maximizing the function $S(\boldsymbol{\theta};\boldsymbol{\theta}')$, giving the following algorithm $\forall l \in \{1,2,\cdots,q-1\}$,

$$\boldsymbol{\theta}_l^{(m+1)} = \boldsymbol{\theta}_l^{(m)} + \frac{2}{q(q-1)}\left[\log\left\{q-\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{r}_i(l)\right\} - \log\left\{q-\mathbb{E}_{\boldsymbol{\theta}^{(m)}}(l)\right\}\right],$$

where

$$\mathbb{E}_{\boldsymbol{\theta}}(l) = \sum_{\boldsymbol{s}}\boldsymbol{s}(l)P(\boldsymbol{s};\boldsymbol{\theta}),\ \forall l \in \{1,2,\cdots,q-1\}.$$

□

Appendix C. QQ-plots of the empirical quantiles of the canonical parameter vector components estimates with respect to their corresponding quantiles of the standard gaussian distribution $\mathcal{N}(0,1)$ for $q = 4$ and $q = 8$ respectively.

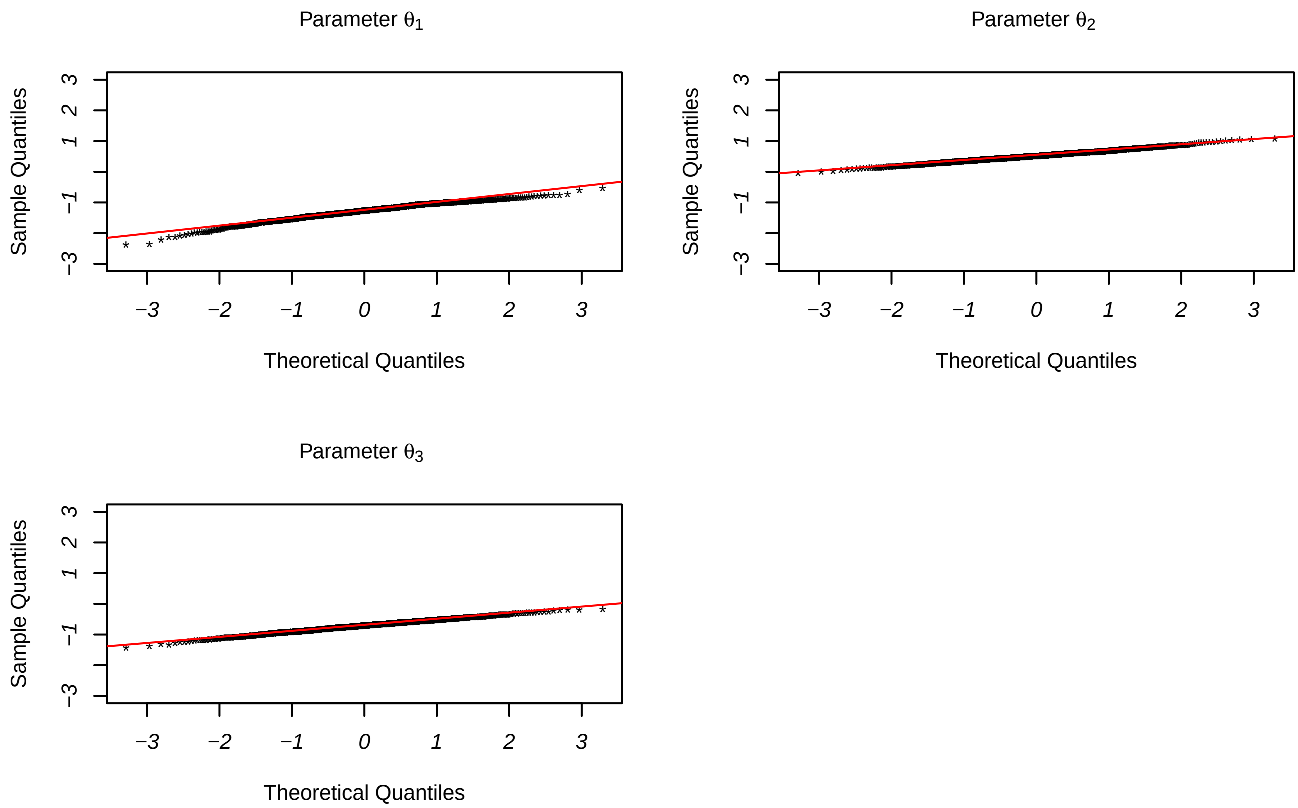

Figure C1. The graphs of quantiles of the sample of each component estimations of the parameter vector for the 1000 generated samples of rankings with common size 100 against the corresponding quantiles of the standard normal distribution $\mathcal{N}(0,1)$ for $q = 4$.

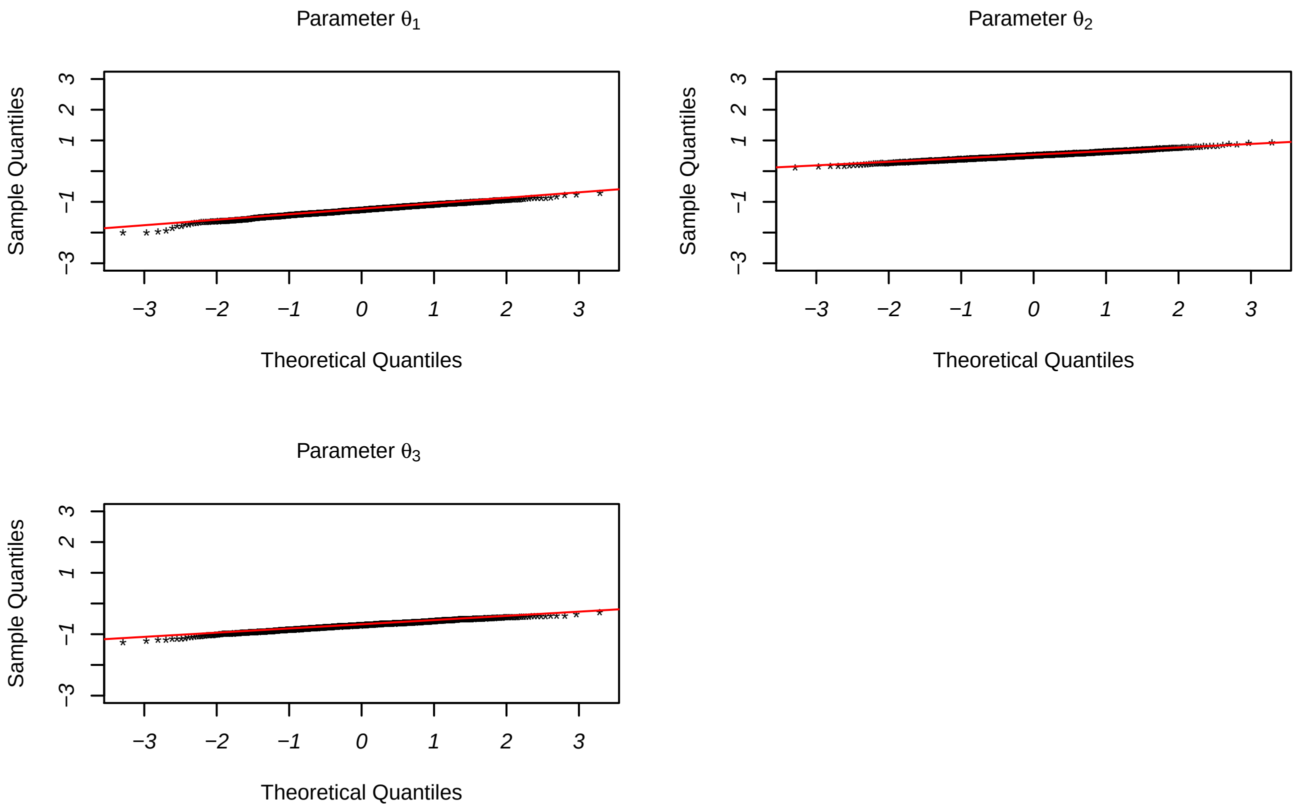

Figure C2. The graphs of quantiles of the sample of each component estimations of the parameter vector for the 1000 generated samples of rankings with common size 200 against the corresponding quantiles of the standard normal distribution $\mathcal{N}(0, 1)$ for $q = 4$.

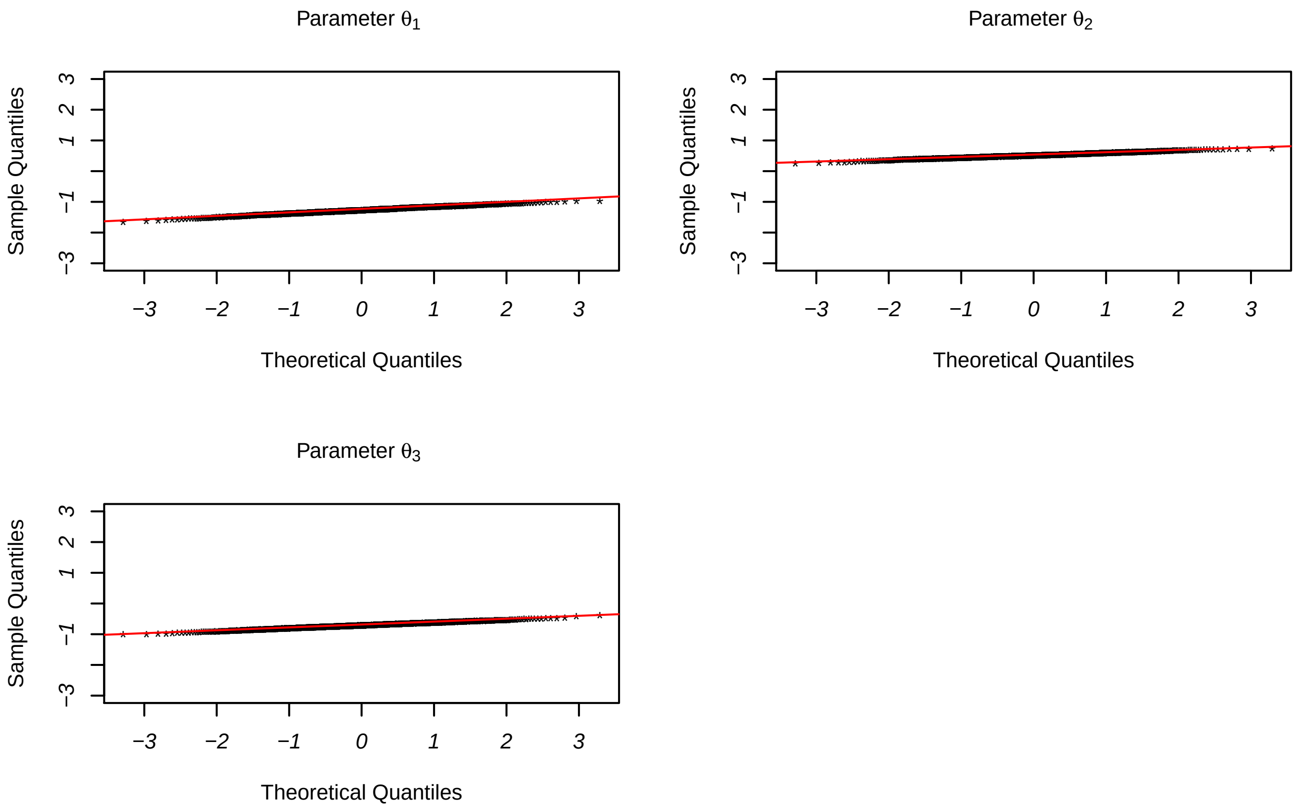

Figure C3. The graphs of quantiles of the sample of each component estimations of the parameter vector for the 1000 generated samples of rankings with common size 500 against the corresponding quantiles of the standard normal distribution $\mathcal{N}(0,1)$ for $q=4$.

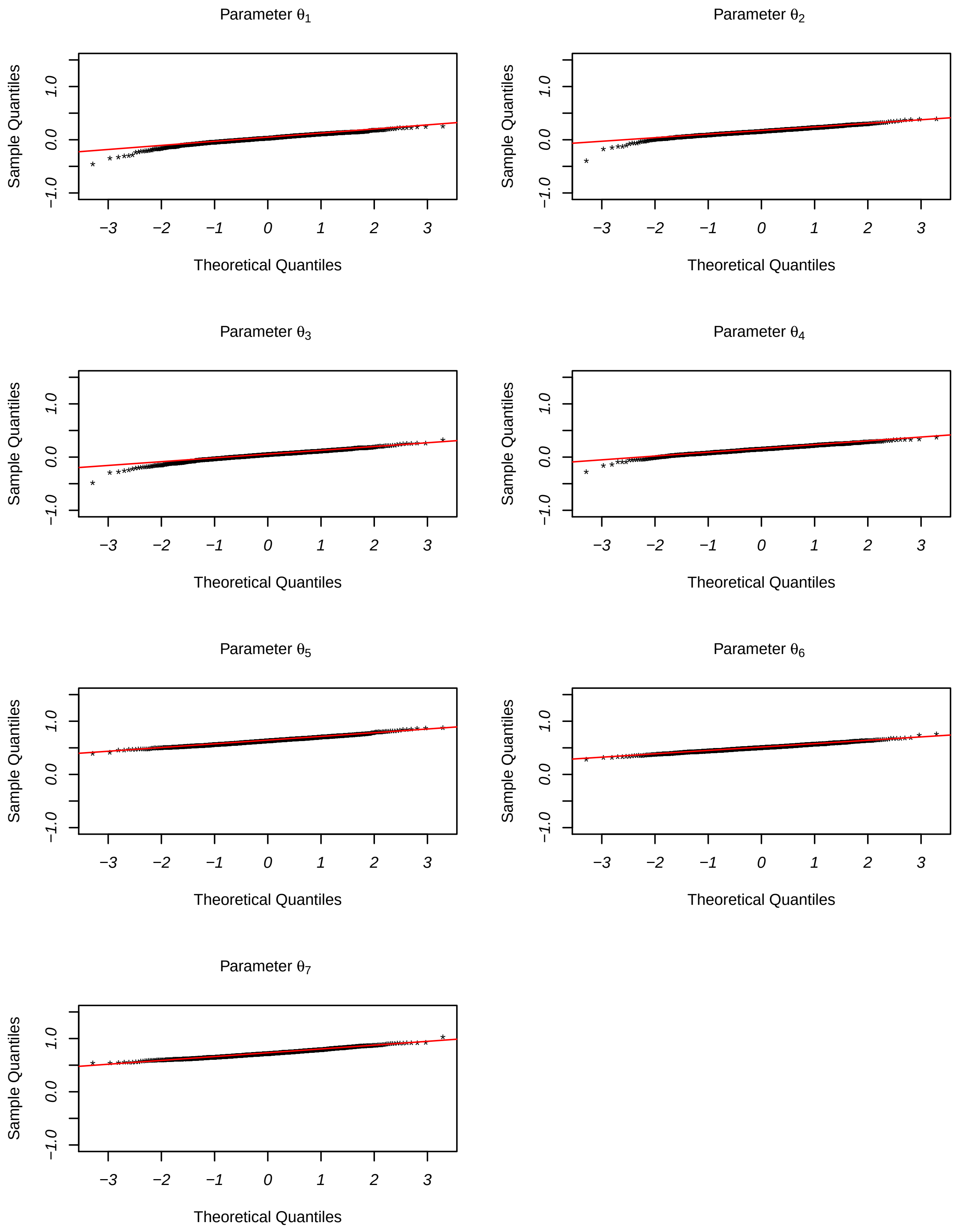

Figure C4. The graphs of quantiles of the sample of each component estimations of the parameter vector for the 1000 generated samples of rankings with common size 100 against the corresponding quantiles of the standard normal distribution $\mathcal{N}(0,1)$ for $q = 8$.

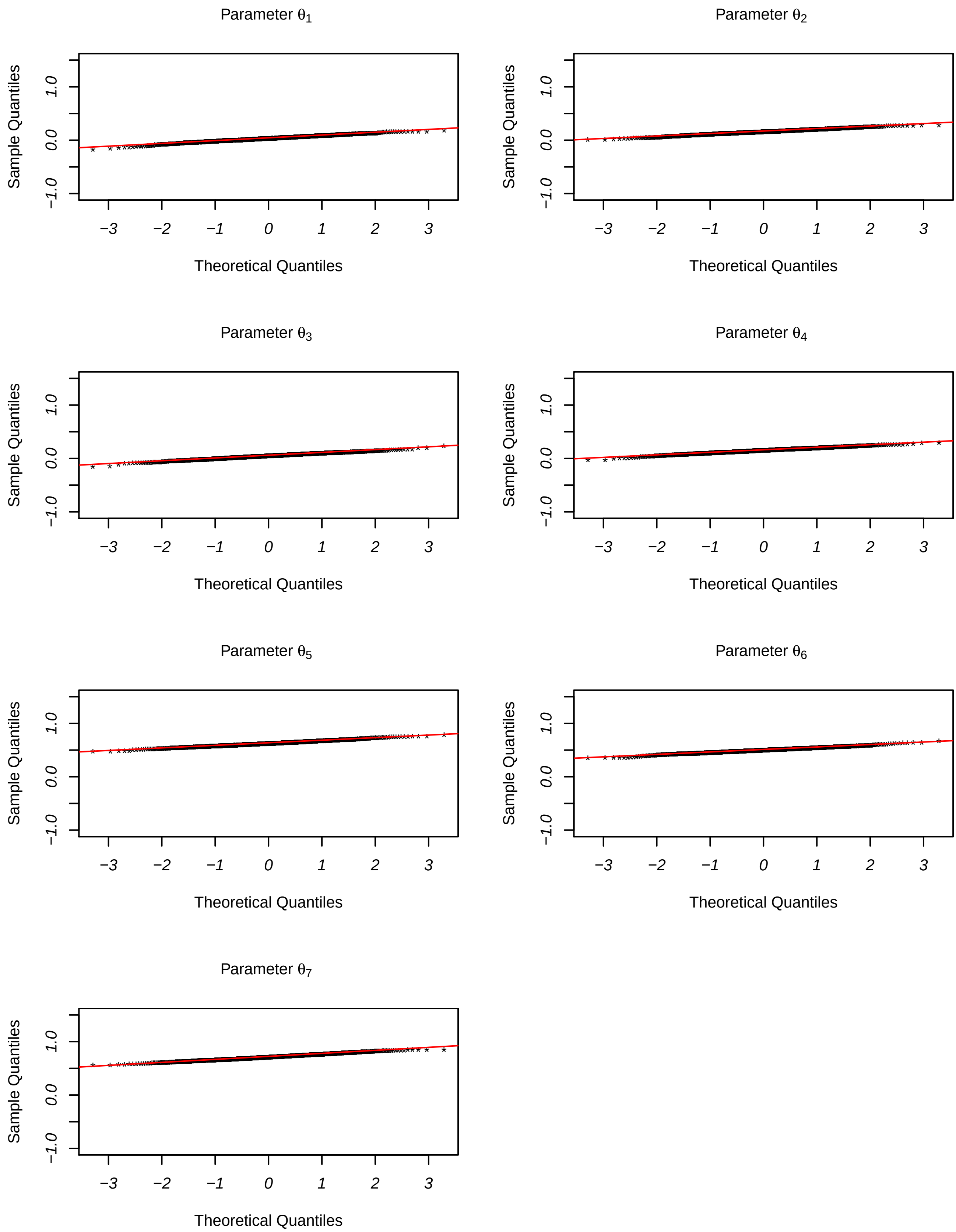

Figure C5. The graphs of quantiles of the sample of each component estimations of the parameter vector for the 1000 generated samples of rankings with common size 200 against the corresponding quantiles of the standard normal distribution $\mathcal{N}(0,1)$ for $q = 8$.

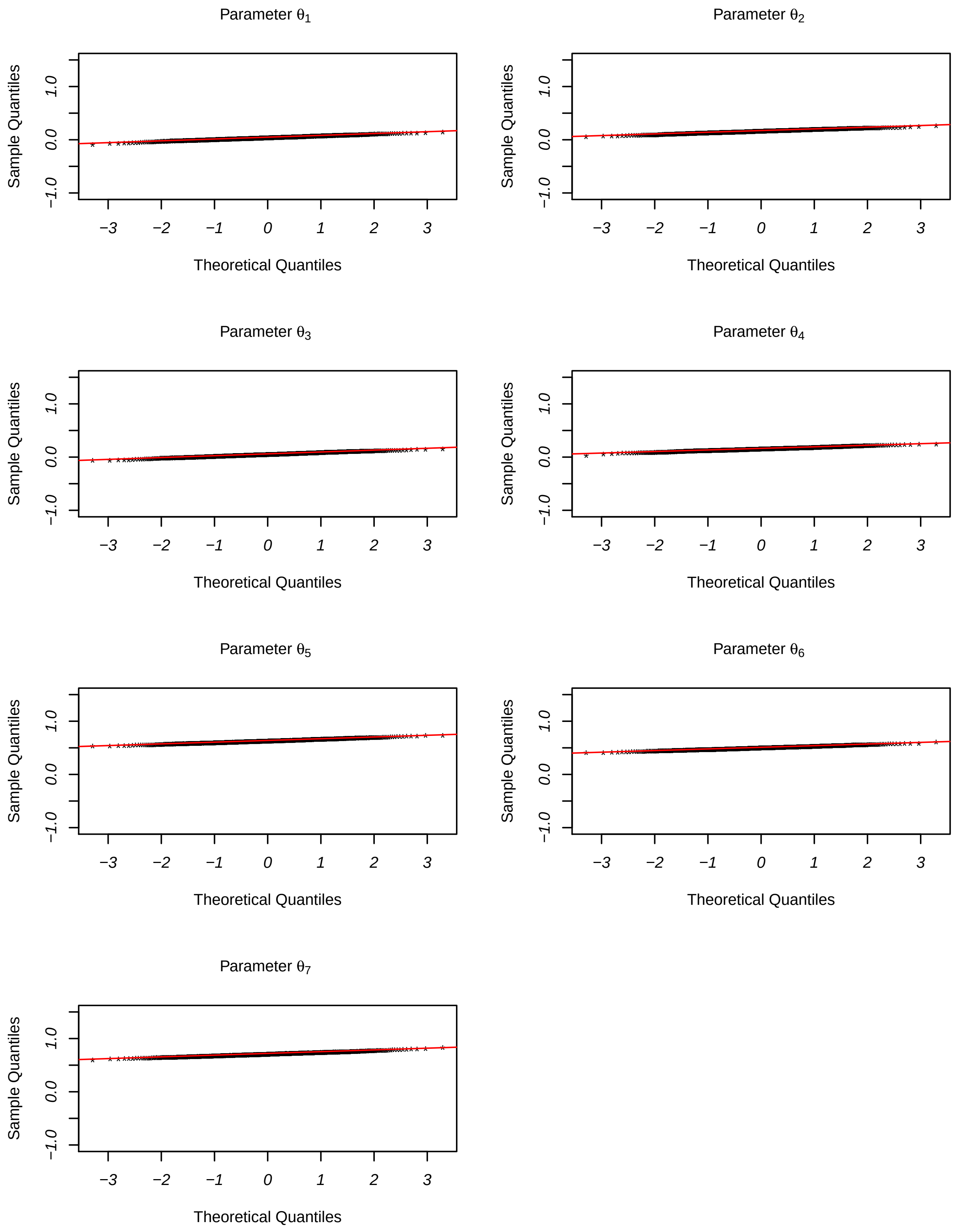

Figure C6. The graphs of quantiles of the sample of each component estimations of the parameter vector for the 1000 generated samples of rankings with common size 500 against the corresponding quantiles of the standard normal distribution $\mathcal{N}(0,1)$ for $q = 8$.